Eight-manifolds with G-structure in eleven dimensional supergravity

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Abstract

We extend the refined G-structure classification of supersymmetric solutions of eleven dimensional supergravity. We derive necessary and sufficient conditions for the existence of an arbitrary number of Killing spinors whose common isotropy group contains a compact factor acting irreducibly in eight spatial dimensions and which embeds in $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$. We use these conditions to explicitly derive the general local bosonic solution of the Killing spinor equation admitting an N=4 SU(4) structure embedding in a $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure, up to an eight-manifold of SU(4) holonomy. Subject to very mild assumptions on the form of the metric, we explicitly derive the general local bosonic solutions of the Killing spinor equation for N=6 Sp(2) structures and N=8 $SU(2) \times SU(2)$ structures embedding in a $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure, again up to eight-manifolds of special holonomy. We construct several other classes of explicit solutions, including some for which the preferred local structure group defined by the Killing spinors does not correspond to any holonomy group in eleven dimensions. We also give a detailed geometrical characterisation of all supersymmetric spacetimes in eleven dimensions admitting G-structures with structure groups of the form $(G \ltimes \mathbb{R}^8) \times \mathbb{R}$.

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1 Introduction

The classification of supersymmetric solutions of supergravity theories has been a long-standing and important problem, due to the central role such spacetimes have played in understanding the physics of string and M-theory. The utility of the notion of G-structures in performing such classifications was first demonstrated in [1]. Since then, G-structures have been used to classify all minimally supersymmetric solutions of several lower dimensional supergravities, for example [2]-[8]; the classification of minimally supersymmetric solutions of eleven dimensional supergravity was given in [9], [10]. Systematic targeted searches have also been made for Minkowski and AdS solutions of string and M-theory, for example, [11]-[18]. These classifications have already spawned many interesting applications, such as the bubbling AdS solutions of [19]; the discovery of an infinite family of Einstein-Sasaki manifolds [20] together with their field theory duals [21], [22]; and the discovery of supersymmetric AdS black holes [23], [24], and supersymmetric black rings [25]-[28]. A review of the G-structure literature is given in [29].

The chief drawback of the original G-structure formalism was that it could only be applied to the classification of spacetimes with minimal supersymmetry. It is known how to classify maximally supersymmetric solutions, using the integrability conditions for the Killing spinor equation [30]. However in [9] it was suggested that G-structure language could be used to systematically classify all supersymmetric solutions admitting any desired number of arbitrary Killing spinors. A universally applicable formalism concretely implementing this proposal was first given in [31], and illustrated in the context of gauged seven dimensional supergravity. The key steps given in this paper are to:

(1) Use a systematic procedure to construct a basis in spinor space by acting on a fiducial spinor with a subset of the Clifford algebra;

(2) Express all Killing spinors in terms of this basis, and choose a basis in spacetime to set them in a simple, canonical form;

(3) Insert the Killing spinors expressed in canonical form into the Killing spinor equation, and use the basis to convert the Killing spinor equation into a set of algebraic conditions on, and relationships between, the spin connection, the fluxes and the first derivatives of the functions defining the Killing spinors.

The notion of a G-structure provides the central organisational principle in implementing this procedure; a set of Killing spinors defines a preferred local G-structure whose structure group is given by the common isotropy group of the Killing spinors.

To maximise the computational efficiency in deriving the conditions for spacetimes

to admit more than one Killing spinor, [31] advocated constructing the basis in spinor space by acting on a *Killing* spinor with a subset of the Clifford algebra. The conditions for the existence of a single arbitrary Killing spinor ϵ may be efficiently computed, as in [31], without having to invoke the Fierzing and bilinears of the original G-structure formalism. Then any other Killing spinor η may be expressed as

$$\eta = Q\epsilon, \tag{1}$$

where Q is some matrix in the relevant subset of the Clifford algebra. Since ϵ is Killing, η is Killing if and only if

$$[\mathcal{D}_\mu, Q]\epsilon = 0, \tag{2}$$

where $\mathcal{D}_\mu$ is the supercovariant derivative. Each spacetime component of the commutator may be expressed as a manifest sum of the basis spinors. The vanishing of the coefficients of every basis spinor in each spacetime component then gives the necessary and sufficient conditions for η to be Killing.

Of particular interest is the application of this formalism to eleven dimensional supergravity. A single Killing spinor in eleven dimensions defines either a timelike or a null Killing vector; the Killing spinor itself is then referred to as timelike or null. The G-structure defined by the Killing spinor has structure group $SU(5)$ or $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$, respectively[1]. Incorporating additional Killing spinors generically breaks the structure group to some subgroup.

This paper is the third in a series, building on the work of [10], in which we are pursuing a systematic cataloguing of the properties of all supersymmetric spacetimes in eleven dimensions, with structure groups embedding in $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$; that is, all supersymmetric spacetimes admitting at least one null Killing spinor. One of the results of [10] was that given the existence of a single null Killing spinor in eleven dimensions, the metric can always be cast in the form

$$ds^2 = 2e^+e^- + \delta_{ij}e^ie^j + (e^9)^2, \tag{3}$$

where

$$\begin{aligned} e^+ &= L^{-1}(du + \lambda), \\ e^- &= dv + \frac{1}{2}\mathcal{F}du + Bdz + \nu, \\ e^9 &= C(dz + \sigma), \\ e^i &= e^i_M dx^M, \end{aligned} \tag{4}$$

[1]More precisely, the existence of a timelike or null Killing spinor at a point implies the existence of a preferred $SU(5)$ or $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure in a neighbourhood of that point.

and where the functions $L, \mathcal{F}, B, C,$ and the one-forms $\lambda, \nu, \sigma, e^i$ are independent of v, and satisfy certain additional conditions. The eight-manifold spanned by the e^i is referred to as the base. Many components of the flux are fixed in terms of the spin connection by the Killing spinor equation for the single null Killing spinor, though some components drop out and are unconstrained. The spin connection for the metric (4) is also computed in [10]. We will use throughout all the results of [10], which are summarised in appendix A, and we adopt all the conventions of that paper.

In [32], an overview of the algebraic aspects of all G-structures associated to Killing spinors in eleven dimensions whose structure groups embed in $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ was given; all such groups were classified, and the spaces of spinors fixed by each were constructed. A basis for spinor space was constructed by acting on a null spinor ϵ, which is required to satisfy the following projections in the null spacetime basis (3):

$$\begin{aligned} \Gamma_{1234}\epsilon = \Gamma_{3456}\epsilon = \Gamma_{5678}\epsilon = \Gamma_{1357}\epsilon &= -\epsilon, \\ \Gamma^{+}\epsilon &= 0. \end{aligned} \tag{5}$$

The spinorial basis is given by

$$\epsilon, \;\; \Gamma^i\epsilon, \;\; \frac{1}{8}J^A_{ij}\Gamma^{ij}\epsilon, \;\; \Gamma^-\epsilon, \;\; \Gamma^{-i}\epsilon, \;\; \frac{1}{8}J^A_{ij}\Gamma^{-ij}\epsilon, \tag{6}$$

where the J^A, $A = 1, ..., 7$ are a set of two-forms defined on the base, which furnish a basis for the **7** of Spin(7); explicit expressions for the J^A are given below. Thus, any additional Killing spinors may be written as

$$\eta = (f + u_i\Gamma^i + \frac{1}{8}f^A J^A_{ij}\Gamma^{ij} + g\Gamma^- + v_i\Gamma^{-i} + \frac{1}{8}g^A J^A_{ij}\Gamma^{-ij})\epsilon, \tag{7}$$

for thirty-two real functions f, u_i, f^A, g, v_i, g^A. By acting with the $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ isotropy group of ϵ, some additional Killing spinors can be simplified, while preserving the constraints on the intrinsic torsion implied by the existence of the Killing spinor ϵ; this essentially amounts to choosing the spacetime basis in such a way that the additional Killing spinors are simplified as much as possible.

In [33], the commutator

$$[\mathcal{D}_\mu, g\Gamma^-]\epsilon, \tag{8}$$

was computed, where for eleven dimensional supergravity,

$$\mathcal{D}_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu\nu\sigma}\Gamma^{\nu\sigma} + \frac{1}{288}(\Gamma_{\mu\nu\sigma\tau\rho} - 8g_{\mu\nu}\Gamma_{\sigma\tau\rho})F^{\nu\sigma\tau\rho}. \tag{9}$$

By imposing $[\mathcal{D}_\mu, g\Gamma^-]\epsilon = 0$, an explicit expression for the general solution of the Killing spinor equation admitting a Spin(7) structure was derived. In this paper, we compute

$$\frac{1}{8}[\mathcal{D}_\mu, f^A J^A_{ij}\Gamma^{ij} + g^A J^A_{ij}\Gamma^{-ij}]\epsilon, \tag{10}$$

and analyse in detail the constraints obtained from imposing

$$[\mathcal{D}_\mu, f + \frac{1}{8} f^A J^A_{ij}\Gamma^{ij} + g\Gamma^- + \frac{1}{8} g^A J^A_{ij}\Gamma^{-ij}]\epsilon = 0, \tag{11}$$

for various choices of multiple additional Killing spinors of the form (7) with $u_i = v_i = 0$. Though we do not provide an exhaustive analysis of all supersymmetric spacetimes admitting Killing spinors of this form, the necessary and sufficient conditions for the existence of any desired number of arbitrary Killing spinors of this type may be read off from our expressions for the commutators. Furthermore, we explicitly derive the *general* local bosonic solution of the Killing spinor equation given the existence of four Killing spinors (at least one of which is null) stabilised by a common $SU(4)$ subgroup of Spin(1,10). We also give a complete geometric characterisation of all supersymmetric spacetimes admitting G-structures with structure group of the form $(G \ltimes \mathbb{R}^8) \times \mathbb{R}$.

In [34], we will compute the commutator

$$[\mathcal{D}_\mu, u_i\Gamma^i + v_i\Gamma^{-i}]\epsilon. \tag{12}$$

This, together with [10], [32], [33] and the present work, will give a complete manual for solving the Killing spinor equation of eleven dimensional supergravity, given the existence of a single null Killing spinor.

Recently, in [35], the method of [31] was reformulated. The authors constructed a basis of timelike Dirac spinors for the analysis of the Killing spinor equation of eleven dimensional supergravity, by acting on a fiducial complex spinor ρ with

$$R_{a_1..a_n}\Gamma^{a_1...a_n}, \tag{13}$$

where the $R_{a_1..a_n}$ furnish a basis for $(0,p)$ forms, $p = 0,...,5$, defined on the ten-dimensional Riemanian base in the timelike basis for eleven dimensional spacetime,

$$ds^2 = -(e^0)^2 + \delta_{ab}e^a e^b. \tag{14}$$

The authors of [35] use slightly more abstract notation, by suppressing the Gamma-matrices in (13) and treating the spinors throughout as forms. By expressing the Killing spinors in terms of this spinorial basis, setting them in a canonical form and inserting in the Killing spinor equation, the conditions for supersymmetry in several particular cases are derived. This same procedure has also been applied to IIB in [36]. Very recently in [37], the action of the eleven dimensional supercovariant derivative on the full spinorial basis (13) has been given. In principle, this result can be used to derive the conditions for supersymmetry for any number of arbitary Killing spinors, whether timelike or null.

However, there are two reasons why the analysis we are pursuing is of relevance. The first is that the timelike spinorial basis (13) is unsuited to the study of null supersymmetry, and in this case it is better to work throughout with a basis constructed

from a null spinor. The second reason concerns the only significant difference between the formalisms of [31] and [35]. Because of the complexity of eleven dimensional supergravity, the expressions for the action of the supercovariant derivative on the full basis of spinors are inevitably complicated. Since we construct the basis (6) by acting on a Killing spinor, we need only compute the commutator $[\mathcal{D}_\mu, Q]\epsilon$. Furthermore, we impose the constraints of [10] for $N = 1$ null supersymmetry on the resulting expression, expressing wherever possible the flux in terms of the spin connection. This radically simplifies the final expression we give for the commutator, and means that our results can be used immediately for reading off the conditions for enhanced supersymmetry. By contrast, since the spinorial basis (13) used in [37] is not constructed by acting on a Killing spinor, the expressions given therein are for

$$\mathcal{D}_\mu(Q\rho) = [\mathcal{D}_\mu, Q]\rho + \{\mathcal{D}_\mu, Q\}\rho, \tag{15}$$

and to use the results for the analysis of enhanced supersymmetry one must first impose the $N = 1$ constraints on the given expressions. Given the inevitable complexity of the results, this is not a computationally trivial task.

Once the conditions for supersymmetry have been computed, it remains to determine which components of the field equations and the Bianchi identity must be imposed on the solution of the Killing spinor equation. We do not undertake a complete analysis of this; rather we assume that the Bianchi identity is always imposed on the solution of the Killing spinor equation (it is unneccesary to make this assumption; however it simplifies the analysis of the integrability condition, at no great practical cost for constructing solutions). The (contracted) integrability condition for ϵ,

$$\Gamma^\nu[\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon = 0, \tag{16}$$

implies that some components of the field equations vanish identically, some satisfy algebraic relationships with one another, and some drop out and are unconstrained. The conditions on the field equations for a single arbitrary null Killing spinor are given in [33]. It is easy to determine which additional field equations are automatically satisfied when one demands the existence of an additional Killing spinor $Q\epsilon$; one simply imposes

$$[\Gamma^\nu[\mathcal{D}_\mu, \mathcal{D}_\nu], Q]\epsilon = 0. \tag{17}$$

The remainder of the paper is organised as follows. In section two, we give the commutator

$$\frac{1}{8}[\mathcal{D}_\mu, f^A J^A_{ij}\Gamma^{ij} + g^A J^A_{ij}\Gamma^{-ij}]\epsilon, \tag{18}$$

imposing the $N = 1$ constraints of [10] on the resulting expression. This calculation, while straightforward, is very long and technical, and requires much manipulation of Spin(7) tensors defined on the base space. We sketch some of the details in appendix A.

In section three, we study particular cases of supersymmetric spacetimes admitting $SU(4)$, $Sp(2)$, $SU(2)\times SU(2)$, $SU(2)$, $U(1)$ or Identity structures. The $SU(2)$ and $U(1)$ structures are of interest, because these preferred local G-structures defined by the Killing spinors have structure groups which do not coincide with any of the possible holonomy groups in eleven dimensions (the $SU(2)$ acts irreducibly in eight dimensions); the presence of non-zero fluxes allows for for supersymmetric spacetimes with previously unrecognised local G-structures defined by the Killing spinors.

We use the commutator of section two to derive the general solution of the Killing spinor equation for an $N=4$ $SU(4)$ structure embedding in a $(Spin(7)\ltimes\mathbb{R}^8)\times\mathbb{R}$ structure, giving the metric, four-form and Killing spinors explicitly, up to an arbitrary eight-manifold of $SU(4)$ holonomy. We find that there are two types of solutions. The first type is very similar to that of [33], since the solution admits a Spin(7) structure in addition to an $SU(4)$ structure. Locally, the first type of solution may always be taken to be as follows. The Killing spinors are given by

$$\epsilon,\ \ \frac{1}{8}J^7_{ij}\Gamma^{ij}\epsilon,\ \ H^{-1/3}(x)\Gamma^{-}\epsilon,\ \ \frac{1}{8}H^{-1/3}(x)J^7_{ij}\Gamma^{-ij}\epsilon, \tag{19}$$

with metric

$$\begin{aligned} ds^2 &= H^{-2/3}(x)\Big(2[du+\lambda(x)_M dx^M][dv+\nu(x)_N dx^N]+[dz+\sigma(x)_M dx^M]^2\Big) \\ &+ H^{1/3}(x)h_{MN}(x)dx^M dx^N, \end{aligned} \tag{20}$$

where h_{MN} is a metric of $SU(4)$ holonomy and $d\lambda$, $d\nu$ and $d\sigma$ are two-forms in the **15** (the adjoint) of $SU(4)$. The four-form is

$$\begin{aligned} F &= e^+\wedge e^-\wedge e^9\wedge d\log H + H^{-1/3}e^+\wedge e^-\wedge d\sigma - e^+\wedge e^9\wedge d\nu \\ &+ H^{-2/3}e^-\wedge e^9\wedge d\lambda + \frac{1}{4!}F^{\mathbf{20}}_{ijkl}e^i\wedge e^j\wedge e^k\wedge e^l, \end{aligned} \tag{21}$$

where $F^{\mathbf{20}}$ denotes the components of F on the base in the selfdual **20** of $SU(4)$. This part of the flux is not fixed by the Killing spinor equation; such flux terms may be used to construct resolved membrane solutions as in [38]. Solutions of this general form have been extensively studied in [39]. The Bianchi identity imposes $F^{\mathbf{20}}=F^{\mathbf{20}}(x)$, $dF^{\mathbf{20}}=0$. One component of the classical four-form field equation must be imposed on the solution of the Killing spinor equation to ensure that all field equations are satisfied; this is

$$\tilde{\nabla}^2 H = -\frac{1}{2}d\sigma_{MN}d\sigma^{MN} - d\lambda_{MN}d\nu^{MN} - \frac{1}{2\times 4!}F^{\mathbf{20}}_{MNPQ}F^{\mathbf{20}MNPQ}, \tag{22}$$

where $\tilde{\nabla}^2$ is the Laplacian on the eight-manifold with metric h_{MN}, and in this equation all indices are raised with h^{MN}.

The second type of solution admitting an $N = 4$ $SU(4)$ structure is determined locally as follows, and as far as we are aware, is new. The Killing spinors are

$$\epsilon,\ \ \frac{1}{8}J^7_{ij}\Gamma^{ij}\epsilon,\ \ (\frac{z}{8}J^7_{ij}\Gamma^{ij}+\cos uH^{-1/3}(x)\Gamma^{-}+\frac{1}{8}\sin uH^{-1/3}(x)J^7_{ij}\Gamma^{-ij})\epsilon,$$

$$(-z-\sin uH^{-1/3}(x)\Gamma^{-}+\frac{1}{8}\cos uH^{-1/3}(x)J^7_{ij}\Gamma^{-ij})\epsilon, \tag{23}$$

with metric

$$ds^2 = H^{-2/3}(x)\Big[2du(dv+\nu(x)_M dx^M)+\cos^2 u dz^2\Big]+H^{1/3}(x)h_{MN}(x)dx^M dx^N, \tag{24}$$

where h_{MN} is a metric of $SU(4)$ holonomy and the one-form ν is required to satisfy

$$d\nu = -\frac{1}{4}HJ + d\nu^{\mathbf{15}}, \tag{25}$$

where J is the complex structure of the Calabi-Yau. The flux is given by

$$\begin{aligned} F &= \cos u(dv+\nu)\wedge du\wedge dz\wedge d(H^{-1})+\cos u du\wedge dz\wedge(-H^{-1}d\nu - J) \\ &+ \frac{1}{4!}F^{\mathbf{20}}_{ijkl}e^i\wedge e^j\wedge e^k\wedge e^l. \end{aligned} \tag{26}$$

Again, the Bianchi identity imposes $F^{\mathbf{20}} = F^{\mathbf{20}}(x)$, $dF^{\mathbf{20}} = 0$. The $+-9$ component of the classical four-form field equation is given by (22) with $\lambda = \sigma = 0$, and all other field equations are identically satisfied. This class of solutions has naked null singularities at $u = \pm\frac{\pi}{2}$; the z direction decompactifies along $u < 0$ before collapsing again along $u > 0$. These two types of solution exhaust all possibilities for $N = 4$ $SU(4)$ structures admitting a null Killing spinor, and together give the general local bosonic solution of the Killing spinor equation for supersymmetric spacetimes in this class. One could, of course, have supersymmetric spacetimes admitting $SU(4)$ structures defined by two or three Killing spinors; however, we do not examine these cases in detail.

Subject to very mild assumptions on the form of the metric[2] we derive the general local bosonic solution of the Killing spinor equation (given the existence of at least one null Killing spinor) for an $N = 6$ $Sp(2)$ structure, an $N = 8$ $SU(2)\times SU(2)$ structure, an $N = 10$ $SU(2)$ structure, and an $N = 12$ $U(1)$ structure, in each case giving the metric, four-form and Killing spinors explicitly, up to eight manifolds of appropriate special holonomy. These solutions all admit an $N = 4$ $SU(4)$ structure of the first type above, and so are contained in that class of solutions. We strongly suspect that there are other solutions analagous to the second type of $N = 4$ $SU(4)$ solutions, which are

[2]Specifically, that the exterior derivative of the one-form ν appearing in (4) lies in the adjoint of the structure group.

excluded by our metric ansatz. It would be straightforward to repeat our exhaustive treatment of $N = 4$ $SU(4)$ structures to determine whether such solutions exist, though we have not done so. Also, one may have $Sp(2)$ structures defined by $N < 6$ Killing spinors, and similarly for the other structure groups; we have not examined such cases in detail. We also find that assuming $d\nu = 0$ in (4), the unique solution of the Killing spinor equation admitting sixteen linearly independent Killing spinors of the form

$$(f + \frac{1}{8} f^A J^A_{ij} \Gamma^{ij} + g\Gamma^- + \frac{1}{8} g^A J^A_{ij} \Gamma^{-ij})\epsilon \tag{27}$$

is the standard asymptotically flat M2 brane. All the supersymmetric spacetimes studied in this section admit both timelike and null Killing spinors.

In section four, we use the commutator to give a complete geometrical characterisation of all supersymmetric spacetimes admitting G-structures with structure groups $(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$, $(Sp(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$, $((SU(2) \times SU(2)) \ltimes \mathbb{R}^8) \times \mathbb{R}$, $(SU(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$, $(U(1) \ltimes \mathbb{R}^8) \times \mathbb{R}$, and some particular classes of $\mathbb{R}^9$ structures. Again, the $(SU(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$ and $(U(1) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structures can only arise for non-zero fluxes. All the supersymmetric spacetimes classified in this section admit only null Killing spinors.

In section 5 we give the integrability conditions for Killing spinors of the form (27), and in section 6 we conclude. The calculation of the commutators is sketched in appendix A, and the technical analysis of $N = 4$ $SU(4)$ structures is relegated to appendix B.

2 The commutator $\frac{1}{8}[\mathcal{D}_\mu, f^A J^A_{ij} \Gamma^{ij} + g^A J^A_{ij} \Gamma^{-ij}]\epsilon$

In this section we compute the commutator which is central to our analysis. As mentioned above, its derivation is very long and technical. We outline some of the steps we use in appendix A. Before giving the result, we will present some introductory material. On an eight-manifold equipped with a Spin(7) structure, tensorial modules of Spin(8) may be decomposed into modules of Spin(7). For forms, these decompositions may be effected by means of certain projectors, a complete set of which is to be found in [10]. However, it will also be convenient for us to work with explicit bases for tensorial modules of Spin(7). Consider first the case of rank two tensors. Under Spin(7), the **28** of Spin(8) decomposes into a **7** and a **21**, with the **1** and the **35** left irreducible. An explicit basis for the **7** is given by the two-forms

$$\begin{aligned}
J^1 &= e^{18} + e^{27} - e^{36} - e^{45}, &\qquad J^2 &= e^{28} - e^{17} - e^{35} + e^{46},\\
J^3 &= e^{38} + e^{47} + e^{16} + e^{25}, &\qquad J^4 &= e^{48} - e^{37} + e^{15} - e^{26}\\
J^5 &= e^{58} + e^{67} - e^{14} - e^{23}, &\qquad J^6 &= e^{68} - e^{57} - e^{13} + e^{24},\\
&& J^7 = e^{78} + e^{56} \quad + \quad & e^{34} + e^{12}.
\end{aligned} \tag{28}$$

Since iJ^A are Gamma-matrices for Spin(7), the J^A obey, with $A = 1, ..., 7$,

$$J^A_{ik} J^{Bk}{}_j = -\delta^{AB}\delta_{ij} + K^{AB}_{ij}, \tag{29}$$

where the K^{AB}_{ij} are antisymmetric on (A, B) and (i, j), and furnish a basis for the **21**. Furthermore,

$$K^{AB}_{ik} J^{Ck}{}_j = T^{ABC}_{ij} + 2\delta^{C[A} J^{B]}_{ij}, \tag{30}$$

where the T^{ABC} are antisymmetric on (ABC), traceless and symmetric on (i, j) and span the **35**. Products of more than three $J^A s$ are related to products of three or less by duality. Since $J^1 J^2 ... J^7 = -1$, defining T^{ABCD} by

$$[T^{ABC}, J^D] = 2T^{ABCD}, \tag{31}$$

we find

$$T^{ABCD} = -\frac{1}{3!}\epsilon^{ABCDEFG} T^{EFG}, \tag{32}$$

with $\epsilon^{1234567} = 1$, and where here, and henceforth unless explicitly indicated otherwide, we adopt the summation convention on Spin(7) indices $A, B, C, ...$

For three-forms, a basis for the **56** of Spin(8) is given by

$$e^i \wedge J^A, \tag{33}$$

so that any three-form P_{ijk} can be written as $3P^A_{[i} J^A_{jk]}$, and is specified by the fifty-six P^A_i. Under Spin(7), $\mathbf{56} \to \mathbf{8} + \mathbf{48}$. The **8** and **48** parts of the form P can be written as

$$\begin{aligned} P^{\mathbf{8}}_{ijk} &= -\frac{3}{7}\phi_{ijk}{}^m P^{Al} J^A_{lm}, \\ P^{\mathbf{48}}_{ijk} &= 3P^A_{[i} J^A_{jk]} + \frac{3}{7}\phi_{ijk}{}^m P^{Al} J^A_{lm}, \end{aligned} \tag{34}$$

where ϕ is the Spin(7) four-form; in our conventions, it is given by

$$\begin{aligned} -\phi &= e^{1234} + e^{1256} + e^{1278} + e^{3456} + e^{3478} + e^{5678} + e^{1357} \\ &+ e^{2468} - e^{1368} - e^{1458} - e^{1467} - e^{2358} - e^{2367} - e^{2457}. \end{aligned} \tag{35}$$

Finally, the space of four-forms decomposes under Spin(7) into $\mathbf{1} + \mathbf{7} + \mathbf{27} + \mathbf{35}$. In our conventions, the **35** is the anti-selfdual part, with the remaining Spin(7) modules being selfdual. Bases for the **1**, **7** and **27** are given respectively by

$$\frac{1}{7} J^A \wedge J^A, \quad K^{AB} \wedge J^B, \quad J^A \wedge J^B - \frac{1}{7}\delta^{AB} J^C \wedge J^C. \tag{36}$$

Note also that $J^A \wedge J^A = -6\phi$. A basis for the **35** is given by

$$K^{[AB} \wedge J^{C]}. \tag{37}$$

We will use these bases extensively in what follows. Now we turn to the expressions for the commutator.

2.1 $[\mathcal{D}_\mu, \frac{1}{8} f^A J^A_{ij}\Gamma^{ij}]\epsilon$

First consider

$$[\mathcal{D}_\mu, \frac{1}{8} f^A J^A_{ij}\Gamma^{ij}]\epsilon. \tag{38}$$

The $-$ component is

$$\Big[\partial_- f^B + \frac{1}{4}(\omega_{-ij} - \frac{1}{3}\omega_{ij-}) f^A K^{ABij}\Big]\frac{1}{8} J^B_{kl}\Gamma^{kl}, \tag{39}$$

where here and below it understood that the given expression acts on ϵ. Next, the $+$ component is

$$\begin{aligned} & f^A\Big[-\frac{4}{7}\omega_{+9i}J^{Ai}{}_j + \frac{1}{4}F^{\mathbf{48}}_{+jkl}J^{Akl}\Big]\Gamma^j + \Big[\partial_+ f^B + \frac{1}{4}(\omega_{+ij} - \frac{1}{2}F_{+9ij}) f^A K^{ABij}\Big]\frac{1}{8}J^B_{kl}\Gamma^{kl} \\ + \quad & \frac{1}{3} f^A J^{Aij}\omega^{\mathbf{7}}_{ij9}\Gamma^- - f^A\Big[\frac{4}{21}(\omega_{i-+} + \omega_{99i})J^{Ai}{}_j + (\omega^{\mathbf{7}}_{[jkl]})^{\mathbf{48}}J^{Akl}\Big]\Gamma^{-j} \\ - \quad & f^A\Big[\frac{1}{48}J^{Aij}F^{\mathbf{27}}_{ijkl}J^{Bkl} + \frac{4}{7}\delta^{AB}\omega_{+9-} + \frac{1}{6}\omega_{ij9}K^{ABij}\Big]\frac{1}{8}J^B_{mn}\Gamma^{-mn}. \end{aligned} \tag{40}$$

The 9 component is

$$\begin{aligned} & \frac{1}{3}\omega^{\mathbf{7}}_{ij9} f^A J^{Aij} - f^A\Big[\frac{8}{21}(\omega_{i-+} + \omega_{99i})J^{Ai}{}_j + 2(\omega^{\mathbf{7}}_{[jkl]})^{\mathbf{48}}J^{Akl}\Big]\Gamma^j \\ + \quad & \Big[\partial_9 f^B + \frac{1}{4}(\omega_{9ij} + \omega_{ij9}) f^A K^{ABij} - f^A\Big(\frac{1}{48}J^{Aij}F^{\mathbf{27}}_{ijkl}J^{Bkl} + \frac{4}{7}\delta^{AB}\omega_{+9-} \\ + \quad & \frac{1}{6}\omega_{ij9}K^{ABij}\Big)\Big]\frac{1}{8}J^B_{mn}\Gamma^{mn} + \frac{1}{6}\omega_{ij-} f^A K^{ABij}\frac{1}{8}J^B_{kl}\Gamma^{-kl}. \end{aligned} \tag{41}$$

The i component is

$$\begin{aligned} - \quad & f^A\Big[\frac{4}{21}(\omega_{j-+} + \omega_{99j})J^{Aj}{}_i + (\omega^{\mathbf{7}}_{[ijk]})^{\mathbf{48}}J^{Ajk}\Big] + f^A\Big[\frac{1}{12}F^{\mathbf{27}}_{ijkl}J^{Akl} + \frac{2}{7}\omega_{+9-}J^A_{ij} \\ - \quad & \frac{2}{3}J^{Ak}{}_{[i}\omega^{\mathbf{21}}_{j]k9} - \frac{1}{6}\delta_{ij}J^{Akl}\omega^{\mathbf{7}}_{kl9} + J^{Ak}{}_{(i}\omega^{\mathbf{35}}_{j)k9}\Big]\Gamma^j + \Big[\partial_i f^B - \frac{1}{2} f^A J^{[A}_{ij}J^{B]}_{kl}\omega^{jkl} \\ + \quad & \frac{1}{4} f^A\Big(\omega_{ijk}K^{ABjk} - (\omega_{99j} + 2\omega_{j-+})K^{ABj}{}_i\Big) + f^A\Big(\frac{4}{21}(\omega_{k-+} + \omega_{99k})J^{Ak}{}_j \\ + \quad & (\omega^{\mathbf{7}}_{[jkl]})^{\mathbf{48}}J^{Akl}\Big)J^{Bj}{}_i\Big]\frac{1}{8}J^B_{mn}\Gamma^{mn} - \frac{2}{3} f^A J^{Ak}_{[i}\omega_{j]k-}\Gamma^{-j} \end{aligned} \tag{42}$$

2.2 $[\mathcal{D}_\mu, \frac{1}{8} g^A J^A_{ij}\Gamma^{-ij}]\epsilon$

Now we compute

$$[\mathcal{D}_\mu, \frac{1}{8} g^A J^A_{ij}\Gamma^{-ij}]\epsilon. \tag{43}$$

In fact, since the spinor ϵ is Killing and, with the choice of spacetime basis (5), constant, we have that

$$\mathcal{D}_\mu(\frac{1}{8}g^A J^A_{ij}\Gamma^{-ij}\epsilon) = [\mathcal{D}_\mu, \frac{1}{8}g^A J^A_{ij}\Gamma^{-ij}]\epsilon = \{\mathcal{D}_\mu, \frac{1}{8}g^A J^A_{ij}\Gamma^{-ij}\}\epsilon. \tag{44}$$

Thus to obtain the action of $\mathcal{D}_\mu$ on $\frac{1}{8}g^A J^A_{ij}\Gamma^{-ij}\epsilon$ we may compute either the commutator or the anticommutator. In this case, it is technically much easier to compute the anticommutator, which is what we have done. The $-$ component is

$$\begin{aligned} & g^A\Big[(-\omega_{-+i} - \frac{5}{7}\omega_{i-+} + \frac{2}{7}\omega_{99i})J^{Ai}{}_j - 2(\omega^{\mathbf{7}}_{[jkl]})^{\mathbf{48}}J^{Akl}\Big]\Gamma^j \\ + \quad & g^A\Big[-(\omega_{-+9} + \frac{1}{7}\omega_{+9-})\delta^{AB} - \frac{1}{24}J^{Aij}F^{\mathbf{27}}_{ijkl}J^{Bkl} + \frac{1}{3}\omega_{ij9}K^{ABij}\Big]\frac{1}{8}J^B_{mn}\Gamma^{mn} \\ + \quad & \Big[\partial_- g^B + \frac{1}{4}(\omega_{-ij} + \omega_{ij-})g^A K^{ABij}\Big]\frac{1}{8}J^B_{kl}\Gamma^{-kl}. \end{aligned} \tag{45}$$

The $+$ component is

$$\begin{aligned} - \quad & \omega_{++i}g^A J^{Ai}{}_j\Gamma^j - \omega_{++9}g^A\frac{1}{8}J^A_{ij}\Gamma^{ij} - \frac{1}{3}\omega_{+ij}g^A J^{Aij}\Gamma^- - g^A\Big[\frac{10}{21}\omega_{+9i}J^{Ai}{}_j \\ + \quad & \frac{1}{12}F^{\mathbf{48}}_{+jkl}J^{Akl}\Big]\Gamma^{-j} + \Big[\partial_+ g^B + \frac{1}{4}(\omega_{+ij} + \frac{1}{6}F_{+9ij})g^A K^{ABij}\Big]\frac{1}{8}J^B_{kl}\Gamma^{-kl}. \end{aligned} \tag{46}$$

The 9 component is

$$\begin{aligned} - \quad & \frac{1}{3}\omega_{+ij}g^A J^{Aij} - g^A\Big[(\omega_{9+i} - \frac{1}{21}\omega_{+9i})J^{Ai}{}_j + \frac{1}{6}F^{\mathbf{48}}_{+jkl}J^{Akl}\Big]\Gamma^j + g^A\Big[\delta^{AB}\omega_{99+} \\ + \quad & \frac{1}{6}F_{+9ij}K^{ABij}\Big]\frac{1}{8}J^B_{kl}\Gamma^{kl} + g^A\Big[-\frac{2}{7}(\omega_{i-+} + \omega_{99i})J^{Ai}{}_j + 2(\omega^{\mathbf{7}}_{[jkl]})^{\mathbf{48}}J^{Akl}\Big]\Gamma^{-j} \\ + \quad & \Big[\partial_9 g^B + \frac{1}{4}(\omega_{9ij} + \omega_{ij9})g^A K^{ABij} + \frac{1}{48}g^A J^{Aij}F^{\mathbf{27}}_{ijkl}J^{Bkl} + \frac{4}{7}g^B\omega_{+9-} \\ - \quad & \frac{1}{6}\omega_{ij9}g^A K^{ABij}\Big]\frac{1}{8}J^B_{mn}\Gamma^{-mn}. \end{aligned} \tag{47}$$

The i component is

$$
\begin{aligned}
- & \quad g^A\Big[\frac{10}{21}\omega_{+9j}J^{Aj}{}_i + \frac{1}{12}F^{\mathbf{48}}_{+ijk}J^{Ajk}\Big] + g^A\Big[J^{Ak}{}_j(\omega_{ik+} + \omega^{\mathbf{7}}_{+ik} + \frac{1}{6}F^{\mathbf{21}}_{+9ik}) \\
+ & \quad \frac{1}{6}J^{Akl}\omega^{\mathbf{7}}_{+kl}\delta_{ij} + \frac{1}{3}J^{Ak}{}_iF^{\mathbf{21}}_{+9jk}\Big]\Gamma^j + g^A\Big[-\omega_{i+9} - \frac{13}{126}\omega_{+9i} - \frac{1}{18}\omega_{+9j}K^{ABj}{}_i \\
- & \quad \frac{17}{144}F^{\mathbf{48}}_{+ijk}K^{ABjk} + \frac{1}{12}J^{(A}_{ij}J^{B)}_{kl}F^{\mathbf{48}jkl}_{+}\Big]\frac{1}{8}J^B_{mn}\Gamma^{mn} + g^A\Big[\frac{1}{2}(\omega_{99j}J^{Aj}{}_i - \omega_{ijk}J^{Ajk}) \\
+ & \quad \Big(-\frac{1}{7}(\omega_{j-+} + \omega_{99j})J^{Aj}{}_i + (\omega^{\mathbf{7}}_{[ijk]})^{\mathbf{48}}J^{Ajk}\Big)\Big]\Gamma^- + g^A\Big[\frac{3}{14}\omega_{+9-}J^A_{ij} - \frac{1}{12}F^{\mathbf{27}}_{ijkl}J^{Akl} \\
- & \quad \frac{2}{3}J^{Ak}{}_{[i}\omega^{\mathbf{21}}_{j]k9} - J^{Ak}{}_{[i}\omega^{\mathbf{35}}_{j]k9}\Big]\Gamma^{-j} + \Big[\partial_i g^B + g^B\omega_{i-+} + \frac{1}{2}g^AJ^{[A}_{ij}J^{B]}_{kl}\omega^{jkl} \\
+ & \quad \frac{1}{4}g^A\Big(\omega_{ijk}K^{ABjk} + (2\omega_{j-+} - \omega_{99j})K^{ABj}{}_i\Big) + g^A\Big(\frac{1}{7}(\omega_{k-+} + \omega_{99k})J^{Ak}{}_j \\
- & \quad (\omega^{\mathbf{7}}_{[jkl]})^{\mathbf{48}}J^{Akl}\Big)J^{Bj}{}_i\Big]\frac{1}{8}J^B_{ij}\Gamma^{-ij}.
\end{aligned}
\tag{48}
$$

2.3 $[\mathcal{D}_\mu, f + g\Gamma^-]\epsilon$

In order to make the discussion of section 3 as self-contained as possible, here we quote the result of [33] for

$$
[\mathcal{D}_\mu, f + g\Gamma^-]\epsilon. \tag{49}
$$

The $-$ component is

$$
\Big[\partial_- f + g(\omega_{+9-} - \omega_{-+9})\Big] + \frac{2g}{3}\omega_{ij9}J^{Aij}\frac{1}{8}J^A_{kl}\Gamma^{kl} + \frac{g}{3}\Big[2\omega_{99i} - \omega_{i-+} - 3\omega_{-+i}\Big]\Gamma^i + \partial_- g\Gamma^-. \tag{50}
$$

The $+$ component is

$$
\Big[\partial_+ f - g\omega_{++9}\Big] - g\omega_{++i}\Gamma^i + \partial_+ g\Gamma^- - \frac{2g}{3}\omega_{+9i}\Gamma^{-i} + \frac{g}{3}\omega_{+ij}J^{Aij}\frac{1}{8}J^A_{kl}\Gamma^{-kl}. \tag{51}
$$

The 9 component is

$$
\begin{aligned}
& \Big[\partial_9 f - g\omega_{9+9}\Big] - \frac{g}{3}\Big[3\omega_{9+i} + \omega_{+9i}\Big]\Gamma^i + \frac{g}{3}\omega_{+ij}J^{Aij}\frac{1}{8}J^A_{kl}\Gamma^{kl} + \partial_9 g\Gamma^- \\
- & \quad \frac{2g}{3}\Big[\omega_{99i} + \omega_{i-+}\Big]\Gamma^{-i} - \frac{g}{3}\omega_{ij9}J^{Aij}\frac{1}{8}J^A_{kl}\Gamma^{-kl}.
\end{aligned}
\tag{52}
$$

The i component is

$$\begin{aligned}
&\Big[\partial_i f + g(-\omega_{i+9} + \frac{1}{3}\omega_{+9i})\Big] + g\Big[\omega_{ij+} - \frac{1}{3}\omega^{\mathbf{7}}_{+ij} + \frac{1}{2}F^{\mathbf{21}}_{+9ij}\Big]\Gamma^j \\
+ \quad & g\Big[\frac{2}{21}\omega_{+9j}J^{Aj}{}_i + \frac{1}{4}F^{\mathbf{48}}_{+ijk}J^{Ajk}\Big]\frac{1}{8}J^A_{mn}\Gamma^{mn} + \Big[\partial_i g + \frac{g}{3}(2\omega_{i-+} - \omega_{99i})\Big]\Gamma^- \\
+ \quad & g\Big[-\frac{1}{2}\delta_{ij}\omega_{+-9} + \frac{4}{3}\omega^{\mathbf{7}}_{ij9} + \omega^{\mathbf{35}}_{ij9}\Big]\Gamma^{-j} + g\Big[-\frac{2}{21}(\omega_{99j} + \omega_{j-+})J^{Aj}{}_i \\
+ \quad & 3(\omega^{\mathbf{7}}_{[ijk]})^{\mathbf{48}}J^{Ajk}\Big]\frac{1}{8}J^A_{mn}\Gamma^{-mn}.
\end{aligned} \tag{53}$$

In deriving the commutators, we have employed the conditions of [10] for $N = 1$ null supersymmetry, which are quoted in Appendix A. Now we will use these expressions to solve the Killing spinor equation in some illustrative special cases.

3 Examples: configurations admitting timelike Killing spinors

In this section, we will consider supersymmetric configurations admitting additional Killing spinors of the form

$$(f + \frac{1}{8}f^A J^A_{ij}\Gamma^{ij} + g\Gamma^- + \frac{1}{8}g^A J^A_{ij}\Gamma^{-ij})\epsilon. \tag{54}$$

It was shown in [32] that provided $g^A \neq 0$, such Killing spinors define a privileged local G-structure which embeds both in an $SU(5)$ and a $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure; that is, some of the Killing spinors are timelike, and some are null. This is what was referred to as a "mixed" G-structure in [32]. In this section, we will examine in detail the conditions obtained by imposing

$$[\mathcal{D}_\mu, f + \frac{1}{8}f^A J^A_{ij}\Gamma^{ij} + g\Gamma^- + \frac{1}{8}g^A J^A_{ij}\Gamma^{-ij}]\epsilon = 0 \tag{55}$$

for various choices of additional Killing spinors. We will explicitly derive the general solution of the Killing spinor equation, given the existence of at least one null Killing spinor, for an $N = 4$ $SU(4)$ structure. We will also obtain explicit solutions for very broad classes of $N = 6$ $Sp(2)$ structures, $N = 8$ $SU(2) \times SU(2)$ structures, $N = 10$ chiral $SU(2)$ structures and $N = 12$ $U(1)$ structures, given very mild assumptions on the form of the metric. We also find that the unique solution of d=11 supergravity admitting sixteen Killing spinors of the form (54) with $\nu = 0$ in (3) is the asymptotically flat M2 brane.

3.1 $N = 4$ $SU(4)$ structures

In [32], it was shown that additional Killing spinors defining an $SU(4)$ structure may taken to be of the form

$$(f + \frac{1}{8} f^7 J^7_{ij} \Gamma^{ij} + g\Gamma^- + \frac{1}{8} g^7 J^7_{ij} \Gamma^{-ij})\epsilon. \tag{56}$$

In appendix B, we derive an explicit form for the metric, four form and Killing spinors of a supersymmetric spacetime admitting the Killing spinor ϵ and three linearly independent spinorial solutions of

$$[\mathcal{D}_\mu, f + \frac{1}{8} f^7 J^7_{ij} \Gamma^{ij} + g\Gamma^- + \frac{1}{8} g^7 J^7_{ij} \Gamma^{-ij}]\epsilon = 0 \tag{57}$$

That is, we classify all supersymmetric spacetimes admitting a preferred local $SU(4)$ structure defined by four Killing spinors, at least one of which is null. To do so, we use the following argument. Every spacetime component of (57) is written, using (39)-(42), (45)-(48) and (50)-(53) as a manifest sum of basis spinors. Derivatives of the functions f, g, f^7, g^7 appear only in the coefficients of the basis spinors ϵ, $\Gamma^-\epsilon$, $\frac{1}{8} J^7_{ij} \Gamma^{ij}\epsilon$, $\frac{1}{8} J^7_{ij} \Gamma^{-ij}\epsilon$, respectively, in each spacetime component. The coefficient of every other basis spinor in each spacetime component is algebraic, and obviously linear, in the functions g, f^7, g^7. Thus the coefficient of each of the twenty-eight basis spinors not involving a derivative of one of the functions f, g, f^7, g^7, in every spacetime component of (57), may be written schematically as

$$f^7 A + gB + g^7 C. \tag{58}$$

Since we are demanding the existence of the Killing spinor ϵ and *three* additional linearly independent Killing spinors of the form (56), we require that $A = B = C = 0$ for the coefficient of each of the twenty-eight basis spinors not involving a derivative in every spacetime component. This will provide a large set of algebraic conditions on, and relationships between, the spin connection and components of the four-form. The remaining conditions imposed by (57) are on the derivatives of the functions f, g, f^7, g^7; given the algebraic constraints, these will simplify considerably. This procedure is worked through in detail in appendix B. Here we will summarise the conditions for supersymmetry derived in the appendix.

Summary The four Killing spinors defining an $N = 4$ $SU(4)$ structure may be taken to be

$$\epsilon, \;\; \frac{1}{8} J^7_{ij} \Gamma^{ij}\epsilon, \;\; \Big(f^7 \frac{1}{8} J^7_{ij} \Gamma^{ij} + g\Gamma^- + g^7 \frac{1}{8} J^7_{ij} \Gamma^{-ij}\Big)\epsilon, \;\; \Big(-f^7 - g^7\Gamma^- + g\frac{1}{8} J^7_{ij} \Gamma^{-ij}\Big)\epsilon. \tag{59}$$

The function g is positive, and f^7, g and g^7 satisfy the following differential equations:

$$\begin{aligned}
\partial_- f^7 = \partial_- g &= \partial_- g^7 = 0, \\
\partial_+ f^7 = 0, \;\; \partial_+ \log g &= \omega_{99+}, \;\; \partial_+ g^7 = -\frac{1}{3} g\omega_{+ij} J^{7ij}, \\
\partial_9 f^7 = -g^7 \omega_{99+} - \frac{1}{3} g\omega_{+ij} J^{7ij}, \qquad & \partial_9 g = \partial_9 g^7 = 0, \\
\partial_i f^7 = 0, \;\; \partial_i \log g &= -\omega_{i-+}, \;\; \partial_i g^7 = -g^7 \omega_{i-+}.
\end{aligned} \tag{60}$$

There are the following algebraic constraints on the non-zero components of the spin connection,

$$\begin{aligned}
g\omega_{99+} &= \frac{1}{3} g^7 \omega_{+ij} J^{7ij}, \\
\omega_{+ij} &= \frac{1}{8} \omega_{+kl} J^{7kl} J^7_{ij} + \omega^{\mathbf{15}}_{+ij}, \\
\omega_{ij+} &= \frac{1}{24} \omega_{+kl} J^{7kl} J^7_{ij} + \omega^{\mathbf{15}}_{ij+}, \\
\omega_{-ij} &= \omega^{\mathbf{15}}_{-ij}, \\
\omega_{ij-} &= \omega^{\mathbf{15}}_{ij-}, \\
\omega_{9ij} &= \omega^{\mathbf{15}}_{9ij}, \\
\omega_{ij9} &= \omega^{\mathbf{15}}_{ij9}, \\
\omega_{99i} &= \omega_{-+i} = -\omega_{i-+} = \omega_{+-i}, \\
\omega^{\mathbf{7}}_{ijk} &= -\frac{1}{4} \delta_{i[j} \partial_{k]} \log g + \frac{1}{8} \phi_{ijk}{}^l \partial_l \log g, \\
\omega^{\mathbf{21}}_{ijk} &= \frac{1}{8} \partial_l \log g K^{7Al}{}_i K^{7A}_{jk} + \omega^{\mathbf{15}}_{ijk},
\end{aligned} \tag{61}$$

and all other components vanish. The only non-zero components of the flux are

$$\begin{aligned}
F_{+-9i} &= 3\omega_{i-+}, \\
F_{+-ij} &= 2\omega^{\mathbf{15}}_{ij9}, \\
F_{+9ij} &= \frac{1}{4} \omega_{+kl} J^{7kl} J^7_{ij} - 2\omega^{\mathbf{15}}_{ij+}, \\
F_{-9ij} &= 2\omega^{\mathbf{15}}_{ij-}, \\
F_{ijkl} &= F^{\mathbf{20}}_{ijkl},
\end{aligned} \tag{62}$$

where $F^{\mathbf{20}}_{ijkl}$ denotes the self-dual **20** piece of F_{ijkl}, $\omega^{\mathbf{15}}_{ij9}$ denotes the **15** piece of $\omega_{[ij]9}$ (similarly for ω_{ij-}, ω_{ij+}), $\omega^{\mathbf{7},\mathbf{21}}_{ijk}$ denotes the projections of ω_{ijk} on the indices j, k onto the **7**, **21** of Spin(7), and $\omega^{\mathbf{15}}_{ijk}$ the projection of ω_{ijk} on the indices j, k onto the **15** of

$SU(4)$. From our discussion of the integrability conditions in section 5, we see that it is sufficient to impose the Bianchi identity for the four-form and the $+-9$ component of the four-form field equation to obtain a solution of all the field equations.

Solving the constraints It is shown in appendix B that all solutions of these constraints fall into two distinct classes, for which the Killing spinors, metric and four-form are given in the introduction. The first class is given by equations (19)-(21), and the second by equations (23)-(26).

3.2 N=6 Sp(2) structures

Now let us consider $N=6$ $Sp(2)$ structures. As was shown in [32], an $N=6$ $Sp(2)$ structure embedding in a $(Spin(7)\ltimes\mathbb{R}^8)\times\mathbb{R}$ structure is defined by ϵ and five additional Killing spinors which may be taken to be of the form

$$(f+g\Gamma^- + \sum_{A=6}^{7}(f^A\frac{1}{8}J^A_{ij}\Gamma^{ij}+g^A\frac{1}{8}J^A_{ij}\Gamma^{-ij}))\epsilon. \tag{63}$$

We may exploit our residual freedom to perform $(Spin(7)\ltimes\mathbb{R}^8)\times\mathbb{R}$ transformations, preserving the form of (63), to take one of the Killing spinors to have $f=g^6=0$. Now we may compute the constraints associated to the existence of these five additional Killing spinors in a completely analagous fashion to the $SU(4)$ case. We have done this with one mild assumption as to the form of the metric: namely that ω_{+ij} contains no singlets under the $Sp(2)$ structure group of the base. With a suitable choice of gauge, this amounts to assuming that the exterior derivative, restricted to the base, of the one-form ν contains no singlets of $Sp(2)$. Note that the analogous assumption in the case of $SU(4)$ would have excluded the solutions of case (iii) in appendix B, equations (23)-(26) of the introduction. We strongly suspect that there are analogous solutions for an $Sp(2)$ structure, but we have not searched for these exhaustively. Given our assumption, we have derived the following necessary and sufficient conditions on the Killing spinors, spin connection and four-form for the existence of an $N=6$ $Sp(2)$ structure. The derivation proceeds exactly as for $SU(4)$. The Killing spinors may be chosen to be

$$\epsilon,\ \ g\Gamma^-\epsilon,\ \ \frac{1}{8}J^7_{ij}\Gamma^{ij}\epsilon,\ \ \frac{1}{8}J^6_{ij}\Gamma^{ij}\epsilon,\ \ \frac{1}{8}gJ^7_{ij}\Gamma^{-ij}\epsilon,\ \ \frac{1}{8}gJ^6_{ij}\Gamma^{-ij}\epsilon, \tag{64}$$

where $g>0$ and

$$\partial_-g=\partial_+g=\partial_9g=0. \tag{65}$$

There are the following constraints on the non-zero components of the spin connection,

$$\begin{aligned}
\omega_{+ij} &= \omega^{\mathbf{10}}_{+ij}, \\
\omega_{ij+} &= \omega^{\mathbf{10}}_{ij+}, \\
\omega_{-ij} &= \omega^{\mathbf{10}}_{-ij}, \\
\omega_{ij-} &= \omega^{\mathbf{10}}_{ij-}, \\
\omega_{9ij} &= \omega^{\mathbf{10}}_{9ij}, \\
\omega_{ij9} &= \omega^{\mathbf{10}}_{ij9}, \\
\omega_{99i} &= \omega_{-+i} = -\omega_{i-+} = \omega_{+-i} = \partial_i \log g, \\
\omega^{\mathbf{7}}_{ijk} &= -\frac{1}{4}\delta_{i[j}\partial_{k]} \log g + \frac{1}{8}\phi_{ijk}{}^{l}\partial_l \log g, \\
\omega^{\mathbf{21}}_{ijk} &= \frac{1}{8}\partial_l \log g K^{7Al}{}_{i}K^{7A}_{jk} + \sum_{A=1}^{5}\frac{1}{8}\partial_l \log g K^{6Al}{}_{i}K^{6A}_{jk} + \omega^{\mathbf{10}}_{ijk},
\end{aligned} \tag{66}$$

and all other components vanish. The only non-zero components of the flux are

$$\begin{aligned}
F_{+-9i} &= 3\omega_{i-+}, \\
F_{+-ij} &= 2\omega^{\mathbf{10}}_{ij9}, \\
F_{+9ij} &= -2\omega^{\mathbf{10}}_{ij+}, \\
F_{-9ij} &= 2\omega^{\mathbf{10}}_{ij-}, \\
F_{ijkl} &= F^{\mathbf{14}}_{ijkl},
\end{aligned} \tag{67}$$

A basis for the **14** is given by

$$J^A \wedge J^B - \frac{1}{5}\delta^{AB}\sum_{1}^{5} J^C \wedge J^C, \quad A, B = 1, ..., 5, \tag{68}$$

while a basis for the **10** is given by

$$K^{AB}, \quad A, B = 1, ..., 5. \tag{69}$$

As was the case for $SU(4)$, an $Sp(2)$ structure of this form embeds in a $Spin(7)$ structure. We may thus take the metric to be given by (20), again with $g = H^{-1/3}$, but now with h_{MN} a metric of $Sp(2)$ holonomy, and $d\lambda$, $d\nu$ and $d\sigma$ restricted to the **10** of $Sp(2)$. Similarly the flux is given by (21), but now with $F_{ijkl} = F^{\mathbf{14}}_{ijkl}$. The Killing spinors define a triplet of complex structures as follows. Let

$$\eta_{(1)} = \frac{1}{8}J^7_{ij}\Gamma^{ij}\epsilon, \quad \eta_{(2)} = \frac{1}{8}J^6_{ij}\Gamma^{ij}\epsilon. \tag{70}$$

Then

$$
\begin{aligned}
\bar{\epsilon}\Gamma_{\mu\nu\sigma}\eta_{(1)} &= -H^{1/3}e^{+}\wedge J^{(1)},\\
\bar{\epsilon}\Gamma_{\mu\nu\sigma}\eta_{(2)} &= -H^{1/3}e^{+}\wedge J^{(2)},\\
\bar{\eta}_{(1)}\Gamma_{\mu\nu\sigma}\eta_{(2)} &= -H^{1/3}e^{+}\wedge J^{(3)},
\end{aligned}
\tag{71}
$$

where

$$
\begin{aligned}
J^{(1)} &= \hat{e}^{12}+\hat{e}^{34}+\hat{e}^{56}+\hat{e}^{78},\\
J^{(2)} &= \hat{e}^{13}+\hat{e}^{42}+\hat{e}^{57}+\hat{e}^{86},\\
J^{(3)} &= \hat{e}^{14}+\hat{e}^{23}+\hat{e}^{58}+\hat{e}^{67},
\end{aligned}
\tag{72}
$$

and the $\hat{e}^i$ are achtbeins for h.

3.3 N=8 $SU(2)\times SU(2)$ structures

An $N=8$ $SU(2)\times SU(2)$ structure is defined by ϵ and seven additional Killing spinors of the form

$$
(f+g\Gamma^{-}+\sum_{A=5}^{7}(f^{A}\frac{1}{8}J^{A}_{ij}\Gamma^{ij}+g^{A}\frac{1}{8}J^{A}_{ij}\Gamma^{-ij}))\epsilon. \tag{73}
$$

We may again exploit our residual freedom to perform $(Spin(7)\ltimes\mathbb{R}^8)\times\mathbb{R}$ transformations to take one of the spinors to have $f=g^5=g^6=0$, and a second to have $g^5=0$. We again expect that there will be two distinct classes of solutions, depending on whether or not $d\nu$ contains singlets of $SU(2)\times SU(2)$. Assuming that it does not, we have derived the general solution of the constraints. The Killing spinors are given by

$$
\epsilon,\ \ g\Gamma^{-}\epsilon,\ \ \frac{1}{8}J^{A}_{ij}\Gamma^{ij}\epsilon,\ \ \frac{1}{8}gJ^{A}_{ij}\Gamma^{-ij}\epsilon,\ \ A=5,6,7. \tag{74}
$$

The metric is given by (20), with $g=H^{-1/3}$, but now h_{MN} is a metric of $SU(2)\times SU(2)$ holonomy, and $d\lambda$, $d\nu$ and $d\sigma$ are in the **6** of $SU(2)\times SU(2)$. The four-form is given by (21), but now $F_{ijkl}=F^{\mathbf{9}}_{ijkl}$, where a basis for the **9** is given by

$$
J^{A}\wedge J^{B}-\frac{1}{4}\delta^{AB}\sum_{1}^{4}J^{C}\wedge J^{C},\ \ A,B=1,...,4, \tag{75}
$$

and a basis for the **6** is given by

$$
K^{AB},\ \ A,B=1,...,4. \tag{76}
$$

We may extract the $SU(2)\times SU(2)$ invariant forms from the spinors in the same way as for $SU(4)$ and $Sp(2)$ structures. Solutions of this form describing membranes on a transverse space of $SU(2)\times SU(2)$ holonomy have been extensively discussed before, for example in [39] and [40].

3.4 N=10 Chiral $SU(2)$ structures

An $N = 10$ "chiral" $SU(2)$ structure is defined by ϵ and nine additional Killing spinors of the form

$$(f + g\Gamma^{-} + \sum_{A=4}^{7}(f^{A}\frac{1}{8}J^{A}_{ij}\Gamma^{ij} + g^{A}\frac{1}{8}J^{A}_{ij}\Gamma^{-ij}))\epsilon. \tag{77}$$

We may again exploit our residual freedom to perform $(Spin(7)\ltimes\mathbb{R}^{8})\times\mathbb{R}$ transformations to take one of the spinors to have $f = g^4 = g^5 = g^6 = 0$, a second to have $g^4 = g^5 = 0$, and a third to have $g^4 = 0$. We again expect that there will be two distinct classes of solutions, depending on whether or not $d\nu$ contains singlets of $SU(2)$. Assuming that it does not, the general solution of the constraints is given as follows. The Killing spinors are

$$\epsilon,\ \ g\Gamma^{-}\epsilon,\ \ \frac{1}{8}J^{A}_{ij}\Gamma^{ij}\epsilon,\ \ \frac{1}{8}gJ^{A}_{ij}\Gamma^{-ij}\epsilon,\ \ A = 4, ..., 7. \tag{78}$$

The metric is again given by (20) with $g = H^{-1/3}$, where $d\lambda$, $d\nu$ and $d\sigma$ are in the $\mathbf{3}$ of $SU(2)$. The flux is given by (21) but with $F_{ijkl} = F^{\mathbf{5}}_{ijkl}$. A basis for the $\mathbf{5}$ is

$$J^{A}\wedge J^{B} - \frac{1}{3}\delta^{AB}\sum_{1}^{3}J^{C}\wedge J^{C},\ \ A, B = 1, 2, 3, \tag{79}$$

and a basis for the $\mathbf{3}$ is given by

$$K^{AB},\ \ A, B = 1, 2, 3. \tag{80}$$

The conformally rescaled base with metric h is required to satisfy

$$\hat{\omega}_{ijk} = \hat{\omega}^{\mathbf{3}}_{ijk}. \tag{81}$$

Thus the conformally rescaled base has holonomy contained in an $SU(2)$ subgroup of Spin(7), an $SU(2)$ subgroup which acts irreducibly in all eight dimensions. By Berger's theorem, the holonomy of the base must thus be the identity, and the base must be locally flat. Though the base is flat, the full eleven dimensional metric for this class of solutions does not in general admit a preferred local identity structure defined by the Killing spinors. This is because the flux and the one-forms λ, ν and σ are incompatible with the existence of such a structure. What is interesting about this class of solutions is that they admit a preferred local geometric structure which is qualitatively different to those previously identified in eleven dimensions. The preferred local structure group defined by the Killing spinors for this class of solutions is strictly $SU(2)$, and not some subgroup, acting irreducibly in eight dimensions. There are no supersymmetric spacetimes in eleven dimensions with strictly $SU(2)$ holonomy, where the $SU(2)$ acts irreducibly in eight dimensions. Hence spacetimes with this sort of local structure defined by the

Killing spinors have (to our knowledge) been overlooked previously. When the flux vanishes, the solutions reduce to $\mathbb{R}^{1,10}$, which of course has holonomy $\{1\}$. Thus the existence of this sort of preferred local geometric structure, associated to a set of Killing spinors, is dependent on non-trivial flux terms being turned on.

3.5 N=12 $U(1)$ structures

An $N = 12$ $U(1)$ structure is defined by ϵ and eleven additional Killing spinors of the form

$$(f + g\Gamma^- + \sum_{A=3}^{7}(f^A\frac{1}{8}J^A_{ij}\Gamma^{ij} + g^A\frac{1}{8}J^A_{ij}\Gamma^{-ij}))\epsilon. \tag{82}$$

Exploiting our residual freedom to perform $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ transformations, we can take one of the spinors to have $f = g^3 = g^4 = g^5 = g^6 = 0$, a second to have $g^3 = g^4 = g^5 = 0$, a third to have $g^3 = g^4 = 0$ and a fourth to have $g^3 = 0$. We assume that $d\nu$ is of the form

$$d\nu = \alpha K^{12}. \tag{83}$$

Then the Killing spinors are

$$\epsilon,\ \ g\Gamma^-\epsilon,\ \ \frac{1}{8}J^A_{ij}\Gamma^{ij}\epsilon,\ \ \frac{1}{8}gJ^A_{ij}\Gamma^{-ij}\epsilon,\ \ A = 3, ..., 7. \tag{84}$$

The metric is again given by (20) with $g = H^{-1/3}$, where $d\lambda$, $d\nu$ and $d\sigma$ are in adjoint of $U(1)$ (that is, they are all proportional to K^{12}), and the conformally rescaled base is locally flat. The flux is given by (21) but with $F_{ijkl} = F^{\mathbf{2}}_{ijkl}$. A basis for the **2** is

$$J^A \wedge J^B - \frac{1}{2}\delta^{AB}\sum_{1}^{2} J^C \wedge J^C,\ \ A, B = 1, 2. \tag{85}$$

As for the chiral $SU(2)$ case, these solutions have a qualitatively new preferred local geometric structure: a $U(1)$ structure acting irreducibly in eight dimensions.

3.6 N=16 Chiral Identity structures

Assuming that $d\nu = 0$ together with the existence of ϵ and fifteen other Killing spinors of the form

$$(f + g\Gamma^- + f^A\frac{1}{8}J^A_{ij}\Gamma^{ij} + g^A\frac{1}{8}J^A_{ij}\Gamma^{-ij})\epsilon, \tag{86}$$

it is straightforward to show, by very similar arguments to those given above, that the general solution of the Killing spinor equation is the standard asymptotically flat M2 brane.

4 Examples: configurations admitting only null Killing spinors

In this section, we will examine in detail all supersymmetric configurations admitting additional Killing spinors of the form

$$(f + \frac{1}{8} f^A J^A_{ij} \Gamma^{ij})\epsilon. \tag{87}$$

It was shown in [32] that the structure group associated to such spinors is of the form $(G \ltimes \mathbb{R}^8) \times \mathbb{R}$, with G a proper subgroup of Spin(7). All these spinors are null, and define what was referred to in [32] as a null G-structure. We now examine in detail the constraints associated with the incorporation of additional supersymmetries of this form.

4.1 $(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structures, N=2

Demanding the existence of a single additional Killing spinor of the form (87) reduces the structure group to $(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$. By acting with the Spin(7) subgroup of the isotropy group of ϵ, the second Killing spinor may be put in the canonical form

$$(f + \frac{1}{8} f^7 J^7_{ij} \Gamma^{ij})\epsilon. \tag{88}$$

The derivation of the constraints is very similar to that of the treatment given to $SU(4)$ structures in appendix B. We find that f and f^7 satisfy

$$\partial_\mu f = \partial_\mu f^7 = 0. \tag{89}$$

By the addition of a constant multiple of ϵ together with a constant rescaling we can take $f = 0$, $f^7 = 1$. Next we find that

$$\begin{aligned} \omega_{ij-} &= \omega^{\mathbf{15}}_{ij-}, \\ \omega_{-ij} &= \omega^{\mathbf{15}}_{-ij}, \\ F_{+9ij} K^{7Aij} &= 2\omega_{+ij} K^{7Aij}. \end{aligned} \tag{90}$$

The additional constraints on ω_{ij9} are

$$\begin{aligned} J^{7ij}\omega_{ij9} &= 0, \\ K^{7Aij}\omega_{ij9} &= -K^{7Aij}\omega_{9ij}, \\ T^{ABCij}\omega_{ij9} &= 0, \quad A, B, C = 1, ..., 6. \end{aligned} \tag{91}$$

Under Spin(6), a symmetric traceless tensor of Spin(7) decomposes as $\mathbf{35} \to \mathbf{20} + \mathbf{15}$. The last of the constraints on ω_{ij9} says that its $\mathbf{20}$ part vanishes. The $\mathbf{27}$ part of F_{ijkl} is given by

$$F^{\mathbf{27}} = -\frac{1}{2}\omega_{+9-}J^7 \wedge J^7 - \frac{3}{7}\omega_{+9-}\phi - \frac{1}{4}K^{7Aij}\omega_{ij9}J^7 \wedge J^A + F^{\mathbf{20}}. \tag{92}$$

The self-dual $\mathbf{20}$ part drops out and is unconstrained. Next consider the constraints on ω_{ijk}. The N=1 constraint

$$\omega_{99i} - 6\omega_{i-+} = -\frac{4}{3}\phi_{ijkl}\omega^{\mathbf{7}jkl} \tag{93}$$

may be rewritten in two equivalent forms,

$$\begin{aligned}(\omega^{\mathbf{7}}_{[ijk]})^{\mathbf{8}} &= \frac{1}{56}\phi_{ijk}{}^{l}(\omega_{99l} - 6\omega_{l-+}), \\ J_i^{Aj}\omega_{jkl}J^{Akl} &= \omega_{99i} - 6\omega_{i-+}.\end{aligned} \tag{94}$$

Rewriting $(\omega^{\mathbf{7}}_{[ijk]})^{\mathbf{48}} = \omega^{\mathbf{7}}_{[ijk]} - (\omega^{\mathbf{7}}_{[ijk]})^{\mathbf{8}}$, the N=2 constraint

$$\frac{4}{21}(\omega_{j-+} + \omega_{99j})J^{7j}{}_{i} + (\omega^{\mathbf{7}}_{[ijk]})^{\mathbf{48}}J^{7jk} = 0 \tag{95}$$

becomes

$$J_i^{7j}\omega_{jkl}J^{7kl} = -(\omega_{99i} + 2\omega_{i-+}). \tag{96}$$

Then from the i component of the Killing spinor equation, we find that

$$\omega_{ijk}K^{7Ajk} = J_i^{7j}J^{Akl}\omega_{jkl}, \quad A \neq 7. \tag{97}$$

Finally we have the condition

$$F^{\mathbf{48}}_{+ijk}J^{7jk} = -\frac{16}{7}\omega_{+9j}J_i^{7j}. \tag{98}$$

Writing

$$F_{+ijk} = 3F^A_{[i}J^A_{jk]}, \tag{99}$$

using the bases for the $\mathbf{8}$ and $\mathbf{48}$ given in the introduction, we find that (98) together with the N=1 constraint

$$F^{\mathbf{8}}_{+ijk} = \frac{2}{7}\phi_{ijk}{}^{l}\omega_{+9l} \tag{100}$$

are equivalent to

$$\begin{aligned}F_i^7 &= -\frac{2}{3}\omega_{+9j}J_i^{7j}, \\ \sum_{A=1}^{6} F^{Aj}J^A_{ij} &= 0.\end{aligned} \tag{101}$$

Summary To summarise, the constraints imposed by the existence of a pair of Killing spinors defining an $(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure, in addition to the N=1 constraints of [10], are as follows. The second Killing spinor may be taken to be

$$\frac{1}{8} J^7_{ij} \Gamma^{ij} \epsilon. \tag{102}$$

The $SU(4)$ invariant almost complex structure and $(4,0)$ form associated to this Killing spinor may be defined as for the $SU(4)$ structures, as is done in appendix B. There are the additional algebraic constraints on the spin connection:

$$\begin{aligned} \omega_{ij-} &= \omega^{\mathbf{15}}_{ij-}, \\ \omega_{-ij} &= \omega^{\mathbf{15}}_{-ij}, \\ J^{7ij}\omega_{ij9} &= 0, \\ K^{7Aij}\omega_{ij9} &= -K^{7Aij}\omega_{9ij}, \\ T^{ABCij}\omega_{ij9} &= 0, \quad A,B,C = 1,...,6, \\ J^{7j}_i \omega_{jkl} J^{7kl} &= -(\omega_{99i} + 2\omega_{i-+}), \\ \omega_{ijk} K^{7Ajk} &= J^{7j}_i J^{Akl} \omega_{jkl}, \quad A \neq 7. \end{aligned} \tag{103}$$

There are the following conditions on the components of the flux not fixed by the N=1 constraints:

$$\begin{aligned} F_{+9ij}K^{7Aij} &= 2\omega_{+ij}K^{7Aij}, \\ F^{\mathbf{27}} &= -\frac{1}{2}\omega_{+9-}J^7 \wedge J^7 - \frac{3}{7}\omega_{+9-}\phi - \frac{1}{4}K^{7Aij}\omega_{ij9}J^7 \wedge J^A + F^{\mathbf{20}}, \\ F^{\mathbf{48}}_{+ijk} &= -2\omega_{+9m}J^{7m}_{[i}J^7_{jk]} - \frac{2}{7}\phi_{ijk}{}^l\omega_{+9l} + 3\sum_{A=1}^{6} F^A_{[i}J^A_{jk]}. \end{aligned} \tag{104}$$

From the N=1 constraints, the forty-eight F^A_i, $A = 1,...,6$ appearing in the definition of $F^{\mathbf{48}}_{+ijk}$ are required to satisfy

$$\sum_{A=1}^{6} F^{Aj} J^A_{ij} = 0. \tag{105}$$

4.2 $(Sp(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structures, N=3

Now assume that there exists a second Killing spinor of the form of the form (87). We may always take this spinor to be of the form

$$(f + f^7 \frac{1}{8} J^7_{ij}\Gamma^{ij} + f^6 \frac{1}{8} J^6_{ij}\Gamma^{ij})\epsilon, \tag{106}$$

and its existence reduces the structure group from $(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$ to $(Sp(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$. Every $(Sp(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure embeds in an $(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure, so the net constraints for an $(Sp(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structure are obtained by adding the constraints for the existence of the Killing spinor (106) to those of the previous subsection. The additional constraints implied by the existence of (106) are as follows. The Killing spinor (106) may be taken to be

$$\frac{1}{8} J^6_{ij} \Gamma^{ij} \epsilon. \tag{107}$$

There are the following algebraic constraints on the spin connection:

$$\begin{aligned}
\omega_{ij-} &= \omega^{\mathbf{10}}_{ij-}, \\
\omega_{-ij} &= \omega^{\mathbf{10}}_{-ij}, \\
J^{6ij} \omega_{ij9} &= K^{67ij} \omega_{ij9} = 0, \\
K^{6Aij} \omega_{ij9} &= -K^{6Aij} \omega_{9ij}, \\
T^{7ABij} \omega_{ij9} &= 0, \quad A, B = 1, ..., 5, \\
J_i^{6j} \omega_{jkl} J^{6kl} &= -(\omega_{99i} + 2\omega_{i-+}), \\
\omega_{ijk} K^{6Ajk} &= J_i^{6j} J^{Akl} \omega_{jkl}, \quad A \neq 6.
\end{aligned} \tag{108}$$

There are the following conditions on the components of the four-form that are not fixed by the N=2 $(SU(4) \ltimes \mathbb{R}^8) \times \mathbb{R}$ constraints:

$$\begin{aligned}
F_{+9ij} K^{6Aij} &= 2\omega_{+ij} K^{6Aij}, \\
F^{\mathbf{20}} &= -\omega_{+9-} \Big(\frac{1}{10} J^7 \wedge J^7 + \frac{3}{5} J^6 \wedge J^6 + \frac{3}{5} \phi \Big) \\
&- \frac{1}{4} \sum_{A=1}^{5} K^{6Aij} \omega_{ij9} J^6 \wedge J^A + F^{\mathbf{14}}, \\
F_i^6 &= -\frac{2}{3} \omega_{+9j} J_i^{6j}.
\end{aligned} \tag{109}$$

4.3 $((SU(2) \times SU(2)) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structures, N=4

Incorporating a third Killing spinor of the form (87) reduces the structure group to $((SU(2) \times SU(2)) \ltimes \mathbb{R}^8) \times \mathbb{R}$. It imposes the following additional constraints. The Killing spinor may be taken to be

$$\frac{1}{8} J^5_{ij} \Gamma^{ij} \epsilon. \tag{110}$$

The spin connection must satisfy the additional conditions

$$\begin{aligned}
\omega_{ij-} &= \omega^{\mathbf{6}}_{ij-}, \\
\omega_{-ij} &= \omega^{\mathbf{6}}_{-ij}, \\
J^{5ij}\omega_{ij9} = K^{57ij}\omega_{ij9} &= K^{56ij}\omega_{ij9} = 0, \\
K^{5Aij}\omega_{ij9} &= -K^{5Aij}\omega_{9ij}, \\
\omega^{\mathbf{35}}_{ij9} &= \frac{1}{8}T^{567kl}\omega_{kl9}T^{567}_{ij}, \\
J^{5j}_{i}\omega_{jkl}J^{5kl} &= -(\omega_{99i} + 2\omega_{i-+}), \\
\omega_{ijk}K^{5Ajk} &= J^{5j}_{i}J^{Akl}\omega_{jkl}, \quad A \neq 5.
\end{aligned} \tag{111}$$

The components of the four-form not fixed by the N=3 $(Sp(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$ constraints satisfy

$$\begin{aligned}
F_{+9ij}K^{5Aij} &= 2\omega_{+ij}K^{5Aij}, \\
F^{\mathbf{14}} &= -\omega_{+9-}\Big(\frac{3}{20}(J^7 \wedge J^7 + J^6 \wedge J^6) + \frac{3}{4}J^5 \wedge J^5 + \frac{9}{10}\phi\Big) \\
&\quad - \frac{1}{4}\sum_{A=1}^{4} K^{5Aij}\omega_{ij9}J^5 \wedge J^A + F^{\mathbf{9}}, \\
F^5_i &= -\frac{2}{3}\omega_{+9j}J^{5j}_{i}.
\end{aligned} \tag{112}$$

4.4 $(SU(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$ structures, N=5

Incorporating a fourth Killing spinor of the form (87) reduces the structure group to $(SU(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$. It imposes the following additional constraints. The Killing spinor may be taken to be

$$\frac{1}{8}J^4_{ij}\Gamma^{ij}\epsilon. \tag{113}$$

The spin connection must satisfy the additional conditions

$$\begin{aligned}
\omega_{ij-} &= \omega^{\mathbf{3}}_{ij-}, \\
\omega_{-ij} &= \omega^{\mathbf{3}}_{-ij}, \\
J^{4ij}\omega_{ij9} = K^{47ij}\omega_{ij9} &= K^{46ij}\omega_{ij9} = K^{45ij}\omega_{ij9} = 0, \\
K^{4Aij}\omega_{ij9} &= -K^{4Aij}\omega_{9ij}, \\
\omega^{\mathbf{35}}_{ij9} &= 0, \\
J^{4j}_{i}\omega_{jkl}J^{4kl} &= -(\omega_{99i} + 2\omega_{i-+}), \\
\omega_{ijk}K^{4Ajk} &= J^{4j}_{i}J^{Akl}\omega_{jkl}, \quad A \neq 4.
\end{aligned} \tag{114}$$

The components of the four-form not fixed by the N=4 $((SU(2)\times SU(2))\ltimes\mathbb{R}^8)\times\mathbb{R}$ constraints satisfy

$$\begin{aligned}
F_{+9ij}K^{4Aij} &= 2\omega_{+ij}K^{4Aij}, \\
F^{\mathbf{9}} &= -\omega_{+9-}\Big(\frac{1}{4}\sum_{A=5}^{7}J^A\wedge J^A + J^4\wedge J^4 + \frac{3}{2}\phi\Big) \\
&- \frac{1}{4}\sum_{A=1}^{3}K^{4Aij}\omega_{ij9}J^4\wedge J^A + F^{\mathbf{5}}, \\
F^4_i &= -\frac{2}{3}\omega_{+9j}J_i^{4j}.
\end{aligned} \tag{115}$$

As was the case for a chiral $SU(2)$ structure, the existence of preferred local G-structures of this form is dependent on non-zero flux terms; the $SU(2)$ factor acts irreducibly in eight dimensions.

4.5 $(U(1)\ltimes\mathbb{R}^8)\times\mathbb{R}$ structures, N=6

With a fifth Killing spinor of the form (87), the structure group reduces to $(U(1)\ltimes\mathbb{R}^8)\times\mathbb{R}$. It imposes the following additional constraints. The Killing spinor may be taken to be

$$\frac{1}{8}J^3_{ij}\Gamma^{ij}\epsilon. \tag{116}$$

The spin connection must satisfy the additional conditions

$$\begin{aligned}
\omega_{ij-} &= \frac{1}{8}K^{12kl}\omega_{kl-}K^{12}_{ij}, \\
\omega_{-ij} &= \frac{1}{8}K^{12kl}\omega_{-kl}K^{12}_{ij}, \\
J^{3ij}\omega_{ij9} &= K^{3Aij}\omega_{ij9}=0, \quad A=4,...,7, \\
K^{3Aij}\omega_{ij9} &= -K^{3Aij}\omega_{9ij}, \\
J_i^{3j}\omega_{jkl}J^{3kl} &= -(\omega_{99i}+2\omega_{i-+}), \\
\omega_{ijk}K^{3Ajk} &= J_i^{3j}J^{Akl}\omega_{jkl}, \quad A\neq 3.
\end{aligned} \tag{117}$$

The components of the four-form not fixed by the N=5 $(SU(2) \ltimes \mathbb{R}^8) \times \mathbb{R}$ constraints satisfy

$$
\begin{aligned}
F_{+9ij}K^{3Aij} &= 2\omega_{+ij}K^{3Aij}, \\
F^{\mathbf{5}} &= -\omega_{+9-}\Big(\frac{1}{2}\sum_{A=4}^{7} J^A \wedge J^A + \frac{3}{2}J^3 \wedge J^3 + 3\phi\Big) \\
&- \frac{1}{4}\sum_{A=1}^{2} K^{3Aij}\omega_{ij9}J^3 \wedge J^A + F^{\mathbf{2}}, \\
F^3_i &= -\frac{2}{3}\omega_{+9j}J^{3j}_i. \qquad (118)
\end{aligned}
$$

Again, the existence of such preferred G-structures is dependent on non-zero flux terms; the $U(1)$ factor acts irreducibly in eight dimensions.

4.6 Chiral $\mathbb{R}^9$ structures, N=7

Incorporating a sixth Killing spinor of the form (87) reduces the structure group to $\mathbb{R}^9$. By an abuse of language, we refer to the associated G-structure as a chiral $\mathbb{R}^9$ structure, since it is defined by spinors of the same chirality on the eight dimensional base. The seventh Killing spinor may be chosen as

$$
\frac{1}{8}J^2_{ij}\Gamma^{ij}\epsilon. \qquad (119)
$$

Its existence imposes the following additional conditions on the spin connection,

$$
\begin{aligned}
\omega_{-ij} &= \omega_{ij-} = 0, \\
\omega^{\mathbf{7}}_{ij9} &= \frac{1}{8}J^{1kl}\omega_{kl9}J^1_{ij}, \\
\omega^{\mathbf{21}}_{ij9} &= -\omega^{\mathbf{21}}_{9ij} = \frac{1}{8}K^{12kl}\omega_{kl9}K^{12}_{ij}, \\
J^{2j}_i\omega_{jkl}J^{2kl} &= -(\omega_{99i} + 2\omega_{i-+}), \\
\omega_{ijk}K^{2Ajk} &= J^{2j}_i J^{Akl}\omega_{jkl}, \quad A \neq 2. \qquad (120)
\end{aligned}
$$

Because there is a condition (94) on the sum $J^{Aj}_i\omega_{jkl}J^{Akl}$ from the N=1 constraints, we also find that

$$
J^{1j}_i\omega_{jkl}J^{1kl} = 7\omega_{99i} + 6\omega_{i-+}. \qquad (121)
$$

The four-form is now completely determined by the geometry. The remaining components are given by

$$\begin{aligned} F^{\mathbf{21}}_{+9ij} &= 2\omega^{\mathbf{21}}_{+ij}, \\ F^{\mathbf{2}} &= \frac{3}{2}\omega_{+9-}(J^1\wedge J^1 - J^2\wedge J^2) - \frac{1}{4}K^{21ij}\omega_{ij9}J^2\wedge J^1, \\ F^2_i &= -\frac{2}{3}\omega_{+9j}J^{2j}_i. \end{aligned} \tag{122}$$

Because there is a condition on the sum $F^{Aj}J^A_{ij}$ from the N=1 constraints, we also find that

$$F^1_i = \frac{10}{3}\omega_{+9j}J^{1j}_i. \tag{123}$$

4.7 Chiral $\mathbb{R}^9$ structures, N=8

Incorporating an eighth Killing spinor of the form (87) leads to the following additional constraints. The spinor may be taken to be

$$\frac{1}{8}J^1_{ij}\Gamma^{ij}\epsilon. \tag{124}$$

It imposes

$$\begin{aligned} \omega_{9ij} = \omega_{ij9} &= \omega_{+9-} = \omega_{+9i} = 0, \\ \omega_{i-+} &= -\omega_{99i}. \end{aligned} \tag{125}$$

Using the conditions on ω_{ijk} we have derived already, we see that

$$\begin{aligned} \omega_{ijk}J^{Ajk} &= J^{Aj}_i\omega_{j-+}, \\ \omega_{ijk}K^{ABjk} &= K^{ABj}_{\ \ i}\omega_{j-+}. \end{aligned} \tag{126}$$

Hence

$$\omega_{ijk} = \delta_{i[j}\omega_{k]-+}. \tag{127}$$

Summary Let us summarise the full set of constraints for an N=8 chiral $\mathbb{R}^9$ structure. The Killing spinors are

$$\epsilon, \ \ \frac{1}{8}J^A_{ij}\Gamma^{ij}\epsilon. \tag{128}$$

There are the following constraints on the spin connection:

$$\begin{aligned} \omega_{(\alpha\beta)-} = \omega_{-9i} = \omega_{i9-} &= \omega_{ij-} = \omega_{-ij} = 0, \\ \omega_{ij9} = \omega_{9ij} &= \omega_{9+-} = \omega_{+9i} = 0, \\ \omega_{i-+} &= -\omega_{99i}, \\ \omega_{ijk} &= \delta_{i[j}\omega_{k]-+}. \end{aligned} \tag{129}$$

The only non-zero components of the flux are

$$\begin{aligned} F_{+-9i} &= 3\omega_{i-+}, \\ F_{+9ij} &= 2\omega_{+ij}. \end{aligned} \tag{130}$$

5 Integrability Conditions

In this section, we will examine the integrability conditions for the Killing spinor equation in detail, to determine which field equations are satisfied identically as a consequence of supersymmetry. We will assume that we always impose the Bianchi identity for the four-form. Then the integrability condition for the Killing spinor equation, contracted with Γ^ν, reads

$$\Gamma^\nu[\mathcal{D}_\mu, \mathcal{D}_\nu]\eta = \Big(E_{\mu\nu}\Gamma^\nu + Q_{\nu\sigma\tau}\Gamma_\mu{}^{\nu\sigma\tau} - 6Q_{\mu\nu\sigma}\Gamma^{\nu\sigma}\Big)\eta = 0, \tag{131}$$

where η is any Killing spinor and the Einstein and four-form field equations are respectively $E_{\mu\nu} = 0$, $Q_{\mu\nu\sigma} = 0$. We may, in very similar fashion to the analysis of the constraints for supersymmetry, rewrite the integrability condition as a manifest sum of basis spinors, to deduce which of the field equations are identically satisfied. Imposing the integrability condition for ϵ, we find that E_{++} and $Q^{\mathbf{21}}_{+ij}$ drop out and are unconstrained. Thus these components of the field equations must be imposed on the solution of the constraints for a single null supersymmetry. There are the following relationships between the components of the field equations:

$$\begin{aligned} E_{+-} = E_{99} &= 12Q_{+-9}, \\ E_{+i} &= 18Q_{+i9}, \\ E_{ij} &= -6Q_{+-9}\delta_{ij}, \end{aligned} \tag{132}$$

and all other components are required to vanish by the integrability condition. Thus when there exists a single null Killing spinor, in addition to imposing the Bianchi identity and $E_{++} = Q_{+ij} = 0$, it is sufficient to impose $Q_{+-9} = Q_{+i9} = 0$ to ensure that all field equations are satisfied.

Now suppose there also exists a Killing spinor

$$(f + \frac{1}{8}f^A J^A_{ij}\Gamma^{ij})\epsilon. \tag{133}$$

Given the integrability condition for ϵ, the integrability condition for this Killing spinor reads

$$[\Gamma^\nu[\mathcal{D}_\mu, \mathcal{D}_\nu], \frac{1}{8}f^A J^A_{ij}\Gamma^{ij}]\epsilon = 0. \tag{134}$$

Given the conditions on the components of the field equations implied by the integrability condition for ϵ, it is easy to verify that the $-$, 9 and i components of (134) vanish identically. Only the $+$ component is nontrivial, and it is

$$-9f^A Q_{+ij} K^{ABij} \frac{1}{8} J^B_{kl} \Gamma^{kl} \epsilon = 0. \tag{135}$$

Thus the existence of the Killing spinor (133) imposes that

$$f^A K^{ABij} Q_{+ij} = 0. \tag{136}$$

Hence we must then still impose the Bianchi identity and $E_{++} = Q_{+9i} = Q_{+ij} = Q_{+-9} = 0$ to guarantee a solution of all the field equations.

Next consider

$$[\Gamma^\nu [\mathcal{D}_\mu, \mathcal{D}_\nu], g\Gamma^-]\epsilon. \tag{137}$$

The $-$ component vanishes identically as a consequence of the integrability condition for ϵ. The $+$ component is

$$g(2E_{++} + 24Q_{+9i}\Gamma^{-i})\epsilon. \tag{138}$$

The 9 component is

$$-24gQ_{+9i}\Gamma^i, \tag{139}$$

while the i component is

$$g(-12Q_{+9i} - 36Q^{\mathbf{21}}_{+ij}\Gamma^j + 12Q_{+9j}J^{Aj}{}_i \frac{1}{8} J^A_{kl}\Gamma^{kl})\epsilon. \tag{140}$$

Hence the existence of the Killing spinor $g\Gamma^-\epsilon$ (which defines a Spin(7) structure) implies that we must only impose the Bianchi identity and $Q_{+-9} = 0$, as was shown in [33]. Finally we will compute

$$[\Gamma^\nu [\mathcal{D}_\mu, \mathcal{D}_\nu], \frac{1}{8} g^A J^A_{ij} \Gamma^{-ij}]\epsilon. \tag{141}$$

The $-$ component vanishes identically. The $+$ component is

$$g^A \Big[2E_{++} \frac{1}{8} J^A_{ij}\Gamma^{ij} + 24Q_{+9i}J^{Ai}{}_j \Gamma^{-j} - 3Q_{+ij}K^{ABij} \frac{1}{8} J^B_{kl} \Gamma^{-kl} \Big]\epsilon. \tag{142}$$

The 9 component is

$$g^A(-24Q_{+9i}J^{Ai}{}_j\Gamma^j + 6Q_{+ij}K^{ABij}\frac{1}{8}J^B_{kl}\Gamma^{kl})\epsilon. \tag{143}$$

The i component is

$$g^A \Big[-12Q_{+9j}J^{Aj}{}_i - 12(Q_{+ik}J^{Ak}{}_j + 2Q_{+jk}J^{Ak}{}_i)\Gamma^j + 12Q_{+9k}J^{Ak}{}_j J^{Bj}{}_i \frac{1}{8} J^B_{lm}\Gamma^{lm} \Big]\epsilon. \tag{144}$$

Equipped with these expressions we may deduce the integrability conditions for any number of arbitrary Killing spinors of the form discussed in this paper.

6 Conclusions

In this work, we have extended the systematic analysis of the Killing spinor equation of eleven dimensional supergravity, using the method of [31]. Ultimately, any supersymmetric spacetime is defined by the fact that its geometry, matter content and Killing spinors collectively solve the Killing spinor equation. The method we are employing allows for the extraction, from the Killing spinor equation, of the first order PDEs completely defining, without redundancy, all supersymmetric spacetimes admitting any number of Killing spinors. The defining equations are expressed in what we believe is the most compact and useful form, as a set of algebraic conditions on, and relations between, the spin connection, the flux components and the first derivatives of the functions defining the Killing spinors.

In this series of papers, we are focussing on eleven dimensional spacetimes admitting at least one null Killing spinor. In particular, we have tried to present our results for the defining first order equations in the most user-friendly form possible, by imposing the conditions for $N = 1$ supersymmetry on our results for the commutators of section 2. The timelike case is covered by [9], [35], and (without imposing the $N = 1$ conditions) by [37]. There is some overlap, since some spacetimes can admit both timelike and null Killing spinors; the conditions under which this can arise are determined in [32].

Of course, giving the defining first order equations is not the same as finding all supersymmetric spacetimes; to do this, one has to integrate the defining equations. We have illustrated in detail how this can be done in some particular cases above, and in appendix B. However, for more generic (or fewer) Killing spinors, integrating the defining equations is a very complicated task. Nonetheless, our results provide what is in a sense the geometrical "DNA" of all supersymmetric spacetimes in eleven dimensions; the method we use naturally exploits the underlying geometrical structure, by producing algebraic conditions on the spin connection and flux components, decomposed into modules of the structure group.

When the calculation of the action of the supercovariant derivative on additional Killing spinors is complete, [34], our results, together with those of [10], will allow for an exhaustive classification of all supersymmetric spacetimes admitting at least one null Killing spinor. However, given the number of distinct choices of linearly independent Killing spinors there will be very many distinct classes of supersymmetric spacetimes in eleven dimensions. Beyond giving a rather formal exhaustive list of defining equations, our results will be of use for highly targeted, yet systematic, searches for particular solutions of special interest. One way of doing this would be to make an ansatz for the Killing spinors; that is, to determine what projections the Killing spinors of the solution of interest should obey, in the spacetime basis (3) (for example, the Killing spinors of the M2 brane obey $\Gamma^{+-9}\eta = \eta$). This could be used to write the Killing spinors in the form of (6), and then the conditions for supersymmetry could be read off from our results.

This procedure is identical in spirit to the "algebraic Killing spinor" technique of [41]. However, using our results in this fashion would involve making no initial ansatz at all for the bosonic fields; the only ansatz made would be for the Killing spinors.

The power of our approach lies in its exhaustive nature, and the explicit form taken by the necessary and sufficient conditions we derive for supersymmetry. It is to be hoped that it will be a useful tool in the construction of supersymmetric solutions, both in eleven dimensional supergravity and elsewhere.

7 Acknowledgements

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A Calculating the commutators

The derivation of the commutators of section two, while entirely straightforward, involves some lengthy computation. Here we give more details on how these expressions are derived. Greek indices take values in $\{+,-,1,...,9\}$, and lower-case Roman indices take values in $\{1,...,8\}$. The supercovariant derivative is given by

$$\mathcal{D}_\mu = \nabla_\mu + \frac{1}{288}(\Gamma_{\mu\nu\alpha\beta\gamma} - 8g_{\mu\nu}\Gamma_{\alpha\beta\gamma})F^{\nu\alpha\beta\gamma}. \tag{145}$$

The commutator

$$[\mathcal{D}_\mu, \frac{1}{8}f^A J^A_{ij}\Gamma^{ij}]\epsilon, \tag{146}$$

in terms of the fluxes and spin connection, is given by

$$\begin{aligned} 8[\mathcal{D}_\mu, \frac{1}{8}f^A J^A_{ij}\Gamma^{ij}]\epsilon &= \partial_\mu f^A J^A_{ij}\Gamma^{ij}\epsilon - 2\omega_{\mu i\nu} f^A J^{Ai}{}_j\Gamma^{\nu j}\epsilon - \frac{1}{3}F_{\mu i\beta\gamma} f^A J^{Ai}{}_j\Gamma^{\beta\gamma j}\epsilon \\ &+ \frac{1}{72}F_{\alpha\beta\gamma\delta} f^A J^A_{\mu j}\Gamma^{\alpha\beta\gamma\delta j}\epsilon - \frac{1}{18}F_{j\beta\gamma\delta} f^A J^{Aj}{}_k\Gamma_\mu{}^{\beta\gamma\delta k}\epsilon. \end{aligned} \tag{147}$$

Our objective is now to reduce this expression to a manifest sum of basis spinors. To do so, we employ the following projections satisfied by ϵ:

$$
\begin{aligned}
\Gamma^{+}\epsilon &= 0, \\
\Gamma^{9}\epsilon &= \epsilon, \\
\Gamma^{+-}\epsilon &= \epsilon, \\
\Gamma^{ij}\epsilon &= \frac{1}{8}J^{Aij}J^{A}_{kl}\Gamma^{kl}\epsilon, \\
\Gamma_{ijk}\epsilon &= -\phi_{ijkl}\Gamma^{l}\epsilon, \\
\Gamma_{ijkl}\epsilon &= \phi_{ijkl}\epsilon + \phi_{[ijk}{}^{m}\Gamma_{l]m}\epsilon, \\
\Gamma_{ijklm}\epsilon &= 5\phi_{[ijkl}\Gamma_{m]}\epsilon,
\end{aligned} \tag{148}
$$

together with

$$
\begin{aligned}
J^{A}_{ik}J^{Bk}{}_{j} &= -\delta^{AB}\delta_{ij} + K^{AB}_{ij}, \\
K^{AB}_{ik}J^{Ck}{}_{j} &= T^{ABC}_{ij} + 2\delta^{C[A}J^{B]}_{ij}.
\end{aligned} \tag{149}
$$

We also employ the fact that ϵ is Killing, using the constraints of [10] to eliminate the fluxes in favour of the spin connection wherever possible. These constraints are as follows. The conditions on the spin connection are

$$
\begin{aligned}
\omega_{(\mu\nu)-} = \omega^{\mathbf{7}}_{ij-} = \omega^{\mathbf{7}}_{-ij} &= \omega_{i9-} = \omega_{-9i} = 0, \\
\omega_{+9-} &= \frac{1}{4}\omega^{i}{}_{i9}, \\
\omega^{\mathbf{7}}_{9ij} &= -\omega^{\mathbf{7}}_{ij9}, \\
(\omega^{\mathbf{7}}_{[ijk]})^{\mathbf{8}} &= \frac{1}{56}\phi_{ijk}{}^{l}(\omega_{99l} - 6\omega_{l-+}).
\end{aligned} \tag{150}
$$

The conditions on the four-form are

$$\begin{aligned}
F_{+-9i} &= 2\omega_{i-+} - \omega_{99i}, \\
F_{+-ij} &= 2\omega_{[ij]9}, \\
F^{\mathbf{7}}_{+9ij} &= 2\omega^{\mathbf{7}}_{+ij}, \\
F^{\mathbf{8}}_{+ijk} &= \frac{2}{7}\phi_{ijk}{}^{l}\omega_{+9l}, \\
F^{\mathbf{7}}_{-9ij} &= 0, \\
F^{\mathbf{21}}_{-9ij} &= 2\omega^{\mathbf{21}}_{ij-}, \\
F_{-ijk} &= 0, \\
F^{\mathbf{8}}_{9ijk} &= \frac{2}{7}\phi_{ijk}{}^{l}(\omega_{99l} + \omega_{l-+}), \\
F^{\mathbf{48}}_{9ijk} &= -12(\omega^{\mathbf{7}}_{[ijk]})^{\mathbf{48}}, \\
F^{\mathbf{1}}_{ijkl} &= \frac{3}{7}\omega_{+9-}\phi_{ijkl}, \\
F^{\mathbf{7}}_{ijkl} &= 2\phi_{[ijk}{}^{m}\omega^{\mathbf{7}}_{l]m9}, \\
F^{\mathbf{35}}_{ijkl} &= 2\phi_{[ijk}{}^{m}\omega^{\mathbf{35}}_{l]m9}.
\end{aligned} \tag{151}$$

The $F^{\mathbf{48}}_{+ijk}$, $F^{\mathbf{21}}_{+9ij}$ and $F^{\mathbf{27}}_{ijkl}$ components of the four-form drop out of the Killing spinor equation for ϵ and are unconstrained by the $N = 1$ constraints.

Now, it is very easy to reduce the $-$ component of (147) to the form (39). So consider the $+$ component, imposing all the projections satisfied by ϵ to reduce it to a manifest sum of basis spinors. The term involving Γ^{i} is given by

$$-\frac{1}{2}F_{+}{}^{jkl}\phi_{ijk}{}^{m}J^{A}_{lm}f^{A}\Gamma^{i}\epsilon - 2\omega_{+9i}f^{A}J^{Ai}{}_{j}\Gamma^{j}\epsilon. \tag{152}$$

Now we may use the identity

$$\phi_{i[jk}{}^{m}\alpha^{\mathbf{7}}_{l]m} = -4\delta_{i[j}\alpha^{\mathbf{7}}_{kl]} - \phi_{jkl}{}^{m}\alpha^{\mathbf{7}}_{im}, \tag{153}$$

together with

$$\phi_{ijkl}\alpha^{kl} = -6\alpha^{\mathbf{7}}_{ij} + 2\alpha^{\mathbf{21}}_{ij}, \tag{154}$$

$$\phi^{ijkl}\alpha^{\mathbf{48}}_{jkl} = 0, \tag{155}$$

$$\phi^{iklm}\phi_{jklm} = 42\delta^{i}_{j}, \tag{156}$$

to obtain the form given in (40). The coefficient of $\Gamma^{-i}\epsilon$ may be treated in a very similar fashion. Analysing the $\Gamma^{ij}\epsilon$ term is straightforward. Next consider the $\Gamma^{-}\epsilon$ term, which is

$$\frac{1}{18}F_{iklm}\phi_{j}{}^{klm}f^{A}J^{Aij}\Gamma^{-}\epsilon. \tag{157}$$

Using

$$\begin{aligned}
\alpha^{\mathbf{27}}_{iklm}\phi_j{}^{klm} &= 0, \\
\phi^{ijmn}\phi_{klmn} &= 12\delta^{ij}_{kl} - 4\phi^{ij}{}_{kl},
\end{aligned} \tag{158}$$

we obtain

$$F_{iklm}\phi_j{}^{klm} = 18\delta_{ij}\omega_{+9-} + 48\omega^{\mathbf{7}}_{ij9} + 12\omega^{\mathbf{35}}_{ij9}, \tag{159}$$

and hence the form given in (40). Finally consider the $\Gamma^{-ij}\epsilon$ term. This is

$$-\frac{1}{12}F_{+-ij}f^A K^{ABij} J^B_{lm}\Gamma^{-lm}\epsilon - \frac{1}{18}f^A J^{Ai}{}_j F_{iklm}\phi^{[klm}{}_n\Gamma^{j]n-}\epsilon. \tag{160}$$

To reduce this to the form given in (40), we use (159), (153) and

$$F_{ijkl}J^{Akl} = -\frac{18}{7}\omega_{+9-}J^A_{ij} + 8J^{Ak}{}_{[i}\omega^{\mathbf{7}}_{j]k9} + 4J^{Ak}{}_{[i}\omega^{\mathbf{35}}_{j]k9} + F^{\mathbf{27}}_{ijkl}J^{Akl}. \tag{161}$$

This completes the derivation of (40). The terms appearing in the 9 component of (147) are very similar, and the derivation of (41) is very much along the same lines.

The analysis of the i component of (147) is particularly involved. Consider first terms with no Gamma matrices; these are

$$\frac{1}{18}F_{9klm}f^A J^A_{ij}\phi^{klmj} - \frac{1}{6}F_{9jlm}f^A J^{Aj}{}_k f^A\phi_i{}^{klm}. \tag{162}$$

We may manipulate these in the same fashion as the terms in (152). To convert the terms involving $\Gamma^{-i}\epsilon$ to the given form, we use (154) and

$$\phi_{ij[k}{}^m\alpha^{\mathbf{7}}_{l]m} = 2(\delta_{j[k}\alpha^{\mathbf{7}}_{l]i} - \delta_{i[k}\alpha^{\mathbf{7}}_{l]j}) + \phi_{kl[i}{}^m\alpha^{\mathbf{7}}_{j]m}. \tag{163}$$

To manipulate the $\Gamma^i\epsilon$ terms, we apply (153) and (163) repeatedly, to convert all terms with F_{ijkl} into the forms appearing on the left-hand sides of equations (159) and (161). We then apply these identities to reduce these terms to the given expression after a long calculation. The analysis of the Γ^{-i} term is straightforward. Finally, for the Γ^{ij} term, we use

$$\phi_{i[j}{}^{lm}\alpha_{k]lm} = -6\alpha^{\mathbf{8}}_{ijk} + \alpha^{\mathbf{48}}_{ijk} - \frac{1}{2}\phi_{jk}{}^{lm}\alpha_{ilm}, \tag{164}$$

together with (149) and the basis for the **48** given at the beginning of section 2.

As discussed in the text, it is much easier technically to compute the anticommutator,

$$\{\mathcal{D}_\mu, \frac{1}{8}g^A J^A_{ij}\Gamma^{-ij}\}\epsilon, \tag{165}$$

rather than the commutator. Nevertheless, it is still a long calculation, with the i component again being particularly involved. To determine the coefficient of the $J^A_{ij}\Gamma^{ij}\epsilon$ term in the i component, we need the additional projection

$$\Gamma_{ijklmn}\epsilon = 5\phi_{[ijkl}\Gamma_{mn]}\epsilon, \tag{166}$$

which together with the projections and Spin(7) identities given above, suffices for the calculation of (165). The computation follows very much the same lines as that of $[\mathcal{D}_\mu, \frac{1}{8}f^A J^A_{ij}\Gamma^{ij}]\epsilon$, though it is significantly longer. Full details of the whole calculation are available on request.

B $N = 4$ $SU(4)$ structures

In this appendix, we work through the derivation of the conditions for $N = 4$ $SU(4)$ supersymmetry, given the existence of a null Killing spinor, and then solve these conditions to obtain the general local bosonic solution of the Killing spinor equation of this class.

B.1 Deriving the constraints

We follow the strategy outlined in the main text. First consider the vanishing of the coefficient of $g\Gamma^{-i}\epsilon$ in the 9 component of (57). This imposes that

$$\omega_{99i} = -\omega_{i-+}. \tag{167}$$

Given this constraint, the vanishing of the coefficient of $g\Gamma^{i}\epsilon$ in the $-$ component implies that

$$\omega_{99i} = \omega_{-+i}. \tag{168}$$

The vanishing of the coefficient of $gJ^A_{ij}\Gamma^{-ij}\epsilon$, $A = 1, ..., 6$, in the i component reads

$$J^{Ajk}(\omega^{\mathbf{7}}_{[ijk]})^{\mathbf{48}} = 0, \quad A = 1, ..., 6. \tag{169}$$

Next, from the coefficient of $g^7\Gamma^{-j}\epsilon$ in the 9 component, we may deduce that

$$J^{7jk}(\omega^{\mathbf{7}}_{[ijk]})^{\mathbf{48}} = 0. \tag{170}$$

This, together with (169), implies that

$$(\omega^{\mathbf{7}}_{[ijk]})^{\mathbf{48}} = 0. \tag{171}$$

Using the conditions of [10] given in appendix A for the existence of a single Killing spinor, (167) and (171) imply that

$$
\begin{aligned}
F_{9ijk} &= 0, && (172)\\
\omega^{\mathbf{7}}_{ijk} &= \frac{1}{4}\delta_{i[j}\omega_{k]-+} - \frac{1}{8}\phi_{ijk}{}^{l}\omega_{l-+}. && (173)
\end{aligned}
$$

The vanishing of the coefficient of $g^7 J^A_{jk}\Gamma^{-jk}\epsilon$, $A = 1, ..., 6$, in the i component imposes

$$
2J^{[7}_{ij}J^{A]}_{kl}\omega^{jkl} + K^{7Ajk}\omega_{ijk} + 3\omega_{j-+}K^{7Aj}{}_{i} = 0. \tag{174}
$$

Using (173), this reduces to

$$
K^{7Ajk}\omega_{ijk} = -\omega_{j-+}K^{7Aj}{}_{i}. \tag{175}
$$

It is easy to verify that the conditions (167), (168), (173) and (175) also imply the vanishing of the coefficients of $f^7\Gamma^{-i}\epsilon$ in the + component, $f^7\Gamma^j\epsilon$ in the 9 component, and $f^7\epsilon$, $g^7\Gamma^-\epsilon$ and $f^7 J^A_{ij}\Gamma^{ij}\epsilon$, $A = 1, ..., 6$, in the i component. Next, from the coefficient of $g\Gamma^{-i}\epsilon$ in the + component, we get

$$
\omega_{+9i} = 0. \tag{176}
$$

From the coefficent of $g\Gamma^i\epsilon$ in the 9 component, we get

$$
\omega_{9+i} = 0, \tag{177}
$$

and from the coefficients of $gJ^A_{ij}\Gamma^{ij}\epsilon$, $A = 1, ..., 6$, in the i component and $f^7\Gamma^i\epsilon$ in the + component, that

$$
F^{\mathbf{48}}_{+ijk} = 0. \tag{178}
$$

From the $N = 1$ constraints we deduce that

$$
F_{+ijk} = 0. \tag{179}
$$

Equations (176), (177) and (178) ensure the vanishing of the coefficents of $g^7\Gamma^{-i}\epsilon$ in the + component, $g^7\Gamma^i\epsilon$ in the 9 and $g^7\epsilon$ and $g^7 J^A_{ij}\Gamma^{ij}\epsilon$, $A = 1, ..., 6$, in the i.

Now look at the constraints on ω_{ij9}, ω_{+-9} and F_{ijkl}. From the coefficient of $g\Gamma^{-j}\epsilon$ in the i component, we find

$$
\omega_{+-9} = \omega^{\mathbf{7}}_{ij9} = \omega^{\mathbf{35}}_{ij9} = 0. \tag{180}
$$

From the coefficient of $f^7\Gamma^j\epsilon$ in the i component, we find

$$
F^{\mathbf{27}}_{ijkl}J^{7kl} = 8J^{7k}{}_{[i}\omega^{\mathbf{21}}_{j]k9}, \tag{181}
$$

while from the coefficient of $g^7\Gamma^{-j}\epsilon$ in the i component,

$$F^{\mathbf{27}}_{ijkl}J^{7kl} = -8J^{7k}{}_{[i}\omega^{\mathbf{21}}_{j]k9}. \tag{182}$$

Hence

$$F^{\mathbf{27}}_{ijkl}J^{7kl} = J^{7k}{}_{[i}\omega^{\mathbf{21}}_{j]k9} = 0. \tag{183}$$

Contracting with J^A, the second equality of (183) becomes

$$K^{7Aij}\omega_{ij9} = 0. \tag{184}$$

Since, from the $N = 1$ constraints, $\omega_{+-9} = -\frac{1}{4}\omega^i{}_{i9}$, (180) and (183) imply that

$$\omega_{ij9} = \omega^{\mathbf{15}}_{[ij]9}, \tag{185}$$

where $\omega^{\mathbf{15}}_{[ij]9}$ denotes the projection on to the **15**, or adjoint, of $Spin(6) \cong SU(4)$. The adjoint is spanned by K^{AB}, $A, B = 1, ..., 6$. To solve the constraints on $F^{\mathbf{27}}$, note that on contracting the basis forms for the **27** given in (36) with J^A, we find that $F^{\mathbf{27}}_{ijkl}J^{Akl}$ are the components of a two-form in the **7**. Writing

$$F^{\mathbf{27}} = f^{AB}(J^A \wedge J^B - \frac{1}{7}\delta^{AB}J^C \wedge J^C), \tag{186}$$

we find on contracting $F^{\mathbf{27}}_{ijkl}J^{7kl}$ with J^A, that

$$\begin{aligned} f^{77} &= \frac{1}{7}f^{AA}, \\ f^{7A} &= 0, \quad A = 1, ..., 6. \end{aligned} \tag{187}$$

Hence

$$F^{\mathbf{27}} = \sum_{A,B=1}^{6} f^{AB}(J^A \wedge J^B - \frac{1}{6}\delta^{AB}\sum_{C=1}^{6} J^C \wedge J^C). \tag{188}$$

Under Spin(6), $\mathbf{27} \to \mathbf{1} + \mathbf{6} + \mathbf{20}$. Equation (188) means that the algebraic constraints impose that the **1** and **6** parts of the decomposition of $F^{\mathbf{27}}$ under Spin(6) vanish. Now, given the constraints (180), (185), (188), and the $N = 1$ constraint $\omega^{\mathbf{7}}_{9ij} = -\omega^{\mathbf{7}}_{ij9}$, the coefficient of $g^7 J^A_{ij}\Gamma^{-ij}\epsilon$, $A = 1, ..., 6$, in the 9 component implies that

$$\omega_{9ij} = \omega^{\mathbf{15}}_{9ij}. \tag{189}$$

Equations (180), (185) and (188) also imply the vanishing of the coefficients of $f^7\Gamma^-\epsilon$ and $f^7 J^A_{ij}\Gamma^{-ij}\epsilon$, $A = 1, ..., 6$, in the + component; of $g^7 J^A_{ij}\Gamma^{ij}\epsilon$, $A = 1, ..., 6$, and $gJ^A_{ij}\Gamma^{ij}\epsilon$ in the − component; and of $f^7\epsilon$ and $gJ^A_{ij}\Gamma^{-ij}\epsilon$, $A = 1, ..., 6$, in the 9 component.

Now look at the constraints on $F^{\mathbf{21}}_{+9ij}$, ω_{+ij} and ω_{ij+}. From the coefficient of $g\Gamma^j\epsilon$ in the i component, we find

$$\omega_{ij+} - \frac{1}{3}\omega^{\mathbf{7}}_{+ij} + \frac{1}{2}F^{\mathbf{21}}_{+9ij} = 0. \tag{190}$$

Hence

$$\begin{aligned} \omega_{(ij)+} &= 0, \\ \omega^{\mathbf{7}}_{ij+} &= \frac{1}{3}\omega^{\mathbf{7}}_{+ij}, \\ F^{\mathbf{21}}_{+9ij} &= -2\omega^{\mathbf{21}}_{ij+}. \end{aligned} \tag{191}$$

From the coefficient of $gJ^A_{ij}\Gamma^{-ij}\epsilon$, $A = 1, ..., 6$, in the $+$ component, we find

$$J^{Aij}\omega_{+ij} = 0, \quad A = 1, ..., 6. \tag{192}$$

From the coefficient of $f^7 J^A_{ij}\Gamma^{ij}\epsilon$, $A = 1, ..., 6$, and of $g^7 J^A_{ij}\Gamma^{-ij}\epsilon$, $A = 1, ..., 6$, in the $+$ component, we find that

$$F_{+9ij}K^{7Aij} = \omega_{+ij}K^{7Aij} = \omega_{ij+}K^{7Aij} = 0, \quad A = 1, ..., 6. \tag{193}$$

Equations (191), (192) and (193) imply the vanishing of the coefficients of $g^7 J^A_{ij}\Gamma^{ij}\epsilon$ and $gJ^A_{ij}\Gamma^{ij}\epsilon$, $A = 1, ..., 6$, in the 9 component, and $g^7\Gamma^i\epsilon$ in the i component.

Next, from the coefficients of $f^7 J^A_{ij}\Gamma^{ij}\epsilon$, $A = 1, ..., 6$, in the $-$, and $f^7 J^A_{ij}\Gamma^{-ij}\epsilon$, $A = 1, ..., 6$, in the 9, we find

$$\begin{aligned} \omega_{-ij} &= \omega^{\mathbf{15}}_{-ij} \\ \omega_{ij-} &= \omega^{\mathbf{15}}_{ij-}. \end{aligned} \tag{194}$$

This also implies the vanishing of the coefficient of $f^7\Gamma^{-j}\epsilon$ in the i, and of $g^7 J^A_{ij}\Gamma^{-ij}\epsilon$, $A = 1, ..., 6$, in the $-$.

Finally, from the coefficient of $g\Gamma^i\epsilon$ in the $+$, we find

$$\omega_{++i} = 0, \tag{195}$$

which implies the vanishing of the coefficient of $g^7\Gamma^i\epsilon$ in the $+$. We have now solved all the algebraic constraints on the spin connection and the flux. It remains to address the differential constraints on the spinor components. Given the algebraic constraints we have found, the Killing spinor equation for any of the three additional $SU(4)$ Killing spinors reduces to the following. The $-$ component is

$$\Big[(\partial_- f - g\omega_{-+9}) + (\partial_- f^7 - g^7\omega_{-+9})\frac{1}{8}J^7_{ij}\Gamma^{ij} + \partial_- g\Gamma^- + \partial_- g^7\frac{1}{8}J^7_{ij}\Gamma^{-ij}\Big]\epsilon = 0. \tag{196}$$

The $+$ component is

$$
\begin{aligned}
& \Big[(\partial_+ f - g\omega_{++9}) + (\partial_+ f^7 - g^7\omega_{++9})\frac{1}{8}J^7_{ij}\Gamma^{ij} + (\partial_+ g - \frac{1}{3}g^7\omega_{+ij}J^{7ij})\Gamma^- \\
+ \quad & (\partial_+ g^7 + \frac{1}{3}g\omega_{+ij}J^{7ij})\frac{1}{8}J^7_{kl}\Gamma^{-kl}\Big]\epsilon = 0.
\end{aligned} \tag{197}
$$

The 9 component is

$$
\begin{aligned}
& \Big[(\partial_9 f + g\omega_{99+} - \frac{1}{3}g^7\omega_{+ij}J^{7ij}) + (\partial_9 f^7 + g^7\omega_{99+} + \frac{1}{3}g\omega_{+ij}J^{7ij})\frac{1}{8}J^7_{kl}\Gamma^{kl} + \partial_9 g\Gamma^- \\
+ \quad & \partial_9 g^7 \frac{1}{8}J^7_{ij}\Gamma^{-ij}\Big]\epsilon = 0.
\end{aligned} \tag{198}
$$

The i component is

$$
\Big[(\partial_i f - g\omega_{i+9}) + (\partial_i f^7 - g^7\omega_{i+9})\frac{1}{8}J^7_{jk}\Gamma^{jk} + (\partial_i g + g\omega_{i-+})\Gamma^- + (\partial_i g^7 + g^7\omega_{i-+})\frac{1}{8}J^7_{jk}\Gamma^{jk}\Big]\epsilon = 0. \tag{199}
$$

We demand that there exist three linearly independent solutions of these equations, in addition to the solution $f = \text{const}$, $f^7 = g = g^7 = 0$. We note that the solution $f^7 = \text{const}$, $f = g = g^7 = 0$ always exists, since its existence imposes no further algebraic restrictions on the spin connection beyond those we have already derived. Thus we may take the second Killing spinor to be

$$
\frac{1}{8}J^7_{ij}\Gamma^{ij}\epsilon. \tag{200}
$$

To find the third and fourth Killing spinors, we exploit the fact that we have some residual freedom to act with the isotropy group of ϵ, while preserving both the $N = 1$ constraints and the algebraic constraints derived above. Specifically, by assumption at least one of the remaining Killing spinors has $g \neq 0$. We act on this spinor with

$$
\frac{f}{g}\Gamma^{+9}, \tag{201}
$$

which is an element of the $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$ isotropy group of ϵ. Thus one of the two remaining Killing spinors may always be chosen to have $f = 0$, $g \neq 0$. Without loss of generality, we may always take $g > 0$. Examining the Killing spinor equation for this spinor we find the further algebraic constraints on the spin connection:

$$
\omega_{-+9} = \omega_{++9} = \omega_{i+9} = 0, \tag{202}
$$

$$
g\omega_{99+} = \frac{1}{3}g^7\omega_{+ij}J^{7ij}. \tag{203}
$$

Given these constraints, the differential equations satisfied by the remaining components of the spinor are as follows.

$$\begin{aligned} \partial_- f^7 = \partial_- g &= \partial_- g^7 = 0, \quad (204) \\ \partial_+ f^7 = 0, \ \ \partial_+ \log g &= \omega_{99+}, \ \ \partial_+ g^7 = -\frac{1}{3} g \omega_{+ij} J^{7ij}, \quad (205) \\ \partial_9 f^7 = -g^7 \omega_{99+} - \frac{1}{3} g \omega_{+ij} J^{7ij}, &\qquad \partial_9 g = \partial_9 g^7 = 0, \quad (206) \\ \partial_i f^7 = 0, \ \ \partial_i \log g &= = -\omega_{i-+}, \ \ \partial_i g^7 = -g^7 \omega_{i-+}. \quad (207) \end{aligned}$$

Finally, given a Killing spinor

$$\Big(f^7 \frac{1}{8} J^7_{ij} \Gamma^{ij} + g \Gamma^- + g^7 \frac{1}{8} J^7_{ij} \Gamma^{-ij} \Big) \epsilon, \quad (208)$$

satisfying (203), (204)-(207), it is easy to verify that the linearly independent spinor

$$\Big(f' + g' \Gamma^- + g^{7\prime} \frac{1}{8} J^7_{ij} \Gamma^{-ij} \Big) \epsilon, \quad (209)$$

where $f' = -f^7$, $g' = -g^7$ and $g^{7\prime} = g$, also satisfies the Killing spinor equation, without imposing any further constraints on the spin connection. Thus we have determined all the constraints on the Killing spinors, geometry and four-form for the existence of an $N = 4$ $SU(4)$ structure embedding in $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$. These conditions are summarised in the main body of the text.

B.2 Solving the constraints

In this subsection, we will solve the conditions derived above for $N = 4$ $SU(4)$ supersymmetry. In what follows, we will employ the coordinates of [10] given in the introduction to derive the general solution of these constraints. Let us now briefly justify this choice. Fixing one of the Killing spinors to have $f = 0$, $g \neq 0$ amounts to a choice of frame in spacetime. Recall that the null Killing vector associated to the Killing spinor ϵ is $K = e^+$. The conditions on the spin connection required for supersymmetry which we have derived with this particular choice of frame imply the following conditions on the Lie derivatives of the basis one-forms:

$$\begin{aligned} \mathcal{L}_K e^+ = \mathcal{L}_K e^- &= \mathcal{L}_K e^9 = 0, \\ \mathcal{L}_K e^i &= -(\omega^{\mathbf{15}}_{ij-} + \omega^{\mathbf{15}}_{-ij}) e^j. \quad (210) \end{aligned}$$

Choosing coordinates (as in the introduction) such that

$$e^+ = \frac{\partial}{\partial v}, \quad (211)$$

imply that the basis one forms e^+, e^- and e^9 must be independent of v with this choice of frame. Now, we exploit the fact that we may always make $SU(4)$ rotations of the e^i, preserving all four Killing spinors, under which the e^i transform according to

$$e^i \to (e^i)' = Q^i{}_j e^j. \tag{212}$$

Since ω_{ij-}, ω_{-ij} belong to the adjoint of $SU(4)$, by performing an $SU(4)$ rotation we may always choose the e^i such that

$$\mathcal{L}_K e^i = 0. \tag{213}$$

Hence we can choose the frame so that the Killing spinors are simplified as above, *and* we may then always introduce the v-independent coordinatisation of this frame given in the introduction.

Let us now derive the general solution of the constraints, to obtain the metric, four-form and Killing spinors explicitly, up to an eight-manifold with $SU(4)$ structure. We will consider three distinct cases in turn, depending on whether or not the functions f^7, g^7 are zero or non-zero. They are:

Case (i): $g^7 = 0$. When $g^7 = 0$, from the first algebraic constraint on the spin connection we find that $\omega_{99+} = 0$. From $\partial_+ g^7 = 0$ we find $\omega_{+ij} J^{7ij} = 0$. Hence $\partial_9 f^7 = 0$, so $f^7 = \text{const}$. By adding a constant multiple of the Killing spinor $J^7_{ij}\Gamma^{ij}\epsilon$ we may take $f^7 = 0$. Thus when $g^7 = 0$, we have the additional constraints

$$g^7 = f^7 = \omega_{99+} = \omega_{+ij} J^{7ij} = 0. \tag{214}$$

The function g must satisfy

$$\begin{aligned} \partial_- g = \partial_+ g &= \partial_9 g = 0, \\ \partial_i \log g &= -\omega_{i-+}, \end{aligned} \tag{215}$$

and the four Killing spinors can be chosen to be

$$\epsilon, \;\; \frac{1}{8} J^7_{ij}\Gamma^{ij}\epsilon, \;\; g\Gamma^-\epsilon, \;\; g\frac{1}{8} J^7_{ij}\Gamma^{-ij}\epsilon. \tag{216}$$

The Killing spinors ϵ and $g\Gamma^-\epsilon$ define a Spin(7) structure. In [33], it was shown that given the existence of a Spin(7) structure, the metric may always be cast in the form

$$\begin{aligned} ds^2 &= H^{-2/3}(x)\Big(2[du + \lambda(x)_M dx^M][dv + \nu(x)_N dx^N] + [dz + \sigma(x)_M dx^M]^2\Big) \\ &+ H^{1/3}(x) h_{MN}(x) dx^M dx^N, \end{aligned} \tag{217}$$

where h_{MN} is a metric of Spin(7) holonomy, $g = H^{-1/3}$, and $d\lambda$, $d\nu$ and $d\sigma$ are two-forms in the **21** of Spin(7). Now, the additional constraints on the spin connection implied by the existence of the $N = 4$ $SU(4)$ structure restrict $d\lambda$, $d\nu$ and $d\sigma$ to the **15** of $SU(4)$ (that is, they are required to be traceless (1,1) forms). Also, the final algebraic constraint on the spin connection reads

$$K^{7Ajk}\hat{\omega}_{ijk} = 0, \tag{218}$$

where $\hat{\omega}_{ijk}$ denotes the spin connection of h. This means that

$$\hat{\omega}_{ijk} = \hat{\omega}^{\mathbf{15}}_{ijk}, \tag{219}$$

and so h is a metric of $SU(4)$ holonomy. With the elfbeins as defined in the introduction, the four-form is given by

$$\begin{aligned} F &= e^+ \wedge e^- \wedge e^9 \wedge d\log H + H^{-1/3} e^+ \wedge e^- \wedge d\sigma - e^+ \wedge e^9 \wedge d\nu \\ &+ H^{-2/3} e^- \wedge e^9 \wedge d\lambda + \frac{1}{4!} F^{\mathbf{20}}_{ijkl} \hat{e}^i \wedge \hat{e}^j \wedge \hat{e}^k \wedge \hat{e}^l. \end{aligned} \tag{220}$$

We may construct the complex structure and holomorphic four-form associated to the $SU(4)$ structure as follows. Let

$$\eta = \frac{1}{8} J^7_{ij} \Gamma^{ij} \epsilon \tag{221}$$

Then

$$(e^+ \wedge J)_{\mu\nu\sigma} = -H^{-1/3} \bar{\epsilon} \Gamma_{\mu\nu\sigma} \eta, \tag{222}$$

where

$$J = H^{-1/3} J^7 = \hat{e}^{12} + \hat{e}^{34} + \hat{e}^{56} + \hat{e}^{78}. \tag{223}$$

The holomorphic four-form is given by

$$e^+ \wedge \Omega_{\mu\nu\sigma\tau\rho} = -H^{-2/3} \bar{\epsilon} \Gamma_{\mu\nu\sigma\tau\rho} \epsilon + i H^{-2/3} \bar{\epsilon} \Gamma_{\mu\nu\sigma\tau\rho} \eta - \frac{1}{2} (e^+ \wedge J \wedge J)_{\mu\nu\sigma\tau\rho}, \tag{224}$$

and it takes the canonical form

$$\Omega = (\hat{e}^1 + i\hat{e}^2)(\hat{e}^3 + i\hat{e}^4)(\hat{e}^5 + i\hat{e}^6)(\hat{e}^7 + i\hat{e}^8). \tag{225}$$

Case (ii): $g^7 \neq 0$, $f^7 = 0$. In this case, from $\partial_9 f = 0$ and the first algebraic constraint on the spin connection, we find $\omega_{99+} = \omega_{+ij} J^{7ij} = 0$. Then from the differential equations for g, g^7 we find that $g = g(x)$, $g^7 = g^7(x)$ and

$$\partial_M \log g = \partial_M \log |g^7|. \tag{226}$$

Here, and throughout, upper case Roman letters denote coordinate indices on the base space. Thus $g = \alpha g^7$, for some non-zero constant α. By taking a linear combination of the third and fourth Killing spinors with constant coefficients, we can construct a pair of Killing spinors with $f^7 = g^7 = 0$, and this case reduces to case (i).

Case (iii): $g^7 \neq 0$**,** $f^7 \neq 0$**.** To treat this case, we need to invert the elfbeins given in the introduction. The inverses are given by

$$\begin{aligned}
e^+ &= \frac{\partial}{\partial v}, \\
e^- &= -\frac{1}{2}L\mathcal{F}\frac{\partial}{\partial v} + L\frac{\partial}{\partial u}, \\
e^9 &= -\frac{B}{C}\frac{\partial}{\partial v} + \frac{1}{C}\frac{\partial}{\partial z}, \\
e^i &= \Big[\Big(\frac{1}{2}\mathcal{F}\lambda_M + B\sigma_M - \nu_M\Big)\frac{\partial}{\partial v} - \lambda_M\frac{\partial}{\partial u} - \sigma_M\frac{\partial}{\partial z} + \frac{\partial}{\partial x^M}\Big]E^{iM},
\end{aligned} \tag{227}$$

where $e^i_M E^{jM} = \delta^{ij}$. Now, from $\omega_{i9+} = \omega_{9+i} = 0$, we find that

$$\sigma = \sigma(z, x). \tag{228}$$

From the differential conditions on f^7, $\partial_- f^7 = \partial_+ f^7 = 0$, we find that

$$f^7 = f^7(z, x). \tag{229}$$

The equation $\partial_i f^7 = 0$ reads

$$\partial_M f^7 = \sigma_M \partial_z f^7. \tag{230}$$

If $f^7 = f^7(x)$ then $f^7 = \text{const}$, we may take $f^7 = 0$ and this case reduces to case (i). Otherwise, consider the change of coordinates

$$z' = f^7(z, x), \tag{231}$$

whereby, given that σ satisfies (230), we may set $\sigma = 0$. Next, from

$$\partial_- g = \partial_- g^7 = \partial_9 g = \partial_9 g^7 = 0, \tag{232}$$

we find that

$$g = g(u, x), \quad g^7 = g^7(u, x). \tag{233}$$

If either $g = g(x)$ or $g^7 = g^7(x)$, then from the expressions for $\partial_+ g$, $\partial_+ g^7$, and the first algebraic constraint on the spin connection, we find that $\omega_{99+} = \omega_{+ij}J^{7ij} = 0$, and we are back to case (i). Thus we require $\partial_u g, \partial_u g^7 \neq 0$. Also from $\omega_{+-9} = \omega_{i-9} = 0$, we find that

$$L = L(u, x), \quad \lambda = \lambda(u, x). \tag{234}$$

From the condition $\partial_i \log g = \partial_i \log |g^7|$, we get

$$\partial_M \log \frac{|g^7|}{g} = \lambda_M \partial_u \log \frac{|g^7|}{g}. \tag{235}$$

If $\partial_u \log \frac{|g^7|}{g} = 0$ then $g = \alpha g^7$, for some constant α, and by taking a linear combination with constant coefficients we can construct a Killing spinor with $g^7 = 0$, and we are back to case (i). Otherwise, define a new coordinate u' as

$$u' = \log \frac{|g^7|}{g}. \tag{236}$$

Making this change of coordinate sets $\lambda = 0$, given that λ satisfies (235). Now consider $\partial_i \log g = -\omega_{i-+}$. Since we have chosen our coordinates so that $\lambda = 0$, this reads

$$\partial_M \log g = -\frac{1}{2}\partial_M \log L. \tag{237}$$

Hence

$$L = \tilde{L}(u')g^{-2}, \tag{238}$$

and by defining a new u coordinate we may set $\tilde{L} = 1$ (though note that this means that equation (236) becomes $g^7 = \tilde{g}^7(u)g$). We may determine the function C as follows. The equation $\partial_+ \log g = \omega_{99+}$ is

$$\partial_u \log g = \partial_u \log C. \tag{239}$$

Thus

$$C = \tilde{C}(z', x)g. \tag{240}$$

Since $\lambda = \sigma = 0$, $\partial_i \log g = \omega_{99i}$ becomes

$$\partial_M \log g = \partial_M \log C, \tag{241}$$

so that

$$C = \tilde{C}(z')g. \tag{242}$$

By defining a new z coordinate we may set $\tilde{C} = 1$ (in the new z coordinate, (231) becomes $f^7 = f^7(z)$). Let us now determine the functions B, $\mathcal{F}$, and the coordinate dependence of the form ν. By making a shift

$$v' = v + \int^z B(u, \hat{z}, x) d\hat{z}, \tag{243}$$

we may set $B = 0$. In this gauge, $\omega_{++9} = 0$ reads

$$\partial_z \mathcal{F} = 0. \tag{244}$$

Thus we may make a second, z-independent, shift of v, to set $\mathcal{F} = 0$ while preserving $B = 0$. In this gauge, given that $\sigma = 0$, $\omega_{i+9} = \omega_{++i} = 0$ implies that

$$\nu = \nu(x). \tag{245}$$

Now look at the expression for $\partial_9 f^7$. With the choices we have made for our coordinates, this reads

$$\partial_z f^7 = \partial_u \tilde{g}^7. \tag{246}$$

Since the right-hand side is a function only of u, and $f^7 = f^7(z)$, we must have

$$f^7 = \alpha z + \beta, \ \ \tilde{g}^7 = \alpha u + \gamma, \tag{247}$$

for some constants α, β, γ. By a constant shift in u and z, we may set $\beta = \gamma = 0$, and by a constant positive rescaling of the Killing spinor together with changing the sign of the coordinates if necessary we may take $\alpha = 1$. Now we insert $g^7 = ug$ into the equation for $\partial_+ g^7$, eliminating $\omega_{+ij}J^{7ij}$ in favour of ω_{99+}. We obtain

$$\partial_u \log g = \partial_u \log(1+u^2)^{-1/2}. \tag{248}$$

Hence

$$g = (1+u^2)^{-1/2}\tilde{g}(x), \tag{249}$$

and we have now determined the Killing spinors completely up to one arbitrary positive function $\tilde{g}(x)$. We have also solved all the constraints on the spin connection except for

$$\begin{aligned}
g\omega_{99+} &= \frac{1}{3}g^7\omega_{+ij}J^{7ij}, \\
\omega_{+ij} &= \frac{1}{8}\omega_{+kl}J^{7kl}J^7_{ij} + \omega^{\mathbf{15}}_{+ij}, \\
\omega_{ij+} &= \frac{1}{24}\omega_{+kl}J^{7kl}J^7_{ij} + \omega^{\mathbf{15}}_{ij+}, \\
\omega_{9ij} &= \omega^{\mathbf{15}}_{9ij}, \\
\omega_{ij9} &= \omega^{\mathbf{15}}_{ij9}, \\
\omega^{\mathbf{7}}_{ijk} &= -\frac{1}{4}\delta_{i[j}\partial_{k]}\log g + \frac{1}{8}\phi_{ijk}{}^{l}\partial_l \log g, \\
\omega^{\mathbf{21}}_{ijk} &= \frac{1}{8}\partial_l \log g K^{7Al}{}_{i}K^{7A}_{jk} + \omega^{\mathbf{15}}_{ijk}.
\end{aligned} \tag{250}$$

Consider first $\omega_{ij9} = \omega^{\mathbf{15}}_{ij9}$. This becomes

$$\Lambda_{(ij)} = 0, \tag{251}$$

where $\Lambda_{ij} = \delta_{ik}(\partial_z e^k)_j$. Then $\omega_{9ij} = \omega^{\mathbf{15}}_{9ij}$ imposes that

$$\Lambda_{ij} = \Lambda^{\mathbf{15}}_{ij}. \tag{252}$$

Since Λ_{ij} is in the adjoint of the structure group of the base, this means that the z dependence of the base is pure gauge, and may be removed by a z dependent $SU(4)$

transformation of the achtbeins, while leaving the Killing spinors and all the other associated constraints invariant. Similarly, the **15** part of $M_{ij} = \delta_{ik}(\partial_u e^k)_j$ is pure gauge, and may be removed by means of a z independent $SU(4)$ transformation. To solve the constraints on ω_{ijk}, conformally rescale the base according to $e^i = \tilde{g}^{-1/2}\hat{e}^i$. Since $\lambda, \sigma = 0$, the constraints on ω_{ijk} become

$$\begin{aligned} \hat{\omega}^{\mathbf{7}}_{ijk} &= 0, \\ K^{7Ajk}\hat{\omega}_{ijk} &= 0, \quad A = 1, ..., 6, \end{aligned} \tag{253}$$

where $\hat{\omega}$ denotes the spin connection of the conformally rescaled base. The conditions (253) impose that the conformally rescaled base must be a Calabi-Yau four-fold for all u. Now,

$$\begin{aligned} \omega_{+ij} &= \frac{1}{8}\omega_{+kl}J^{7kl}J^7_{ij} + \omega^{\mathbf{15}}_{+ij}, \\ \omega_{ij+} &= \frac{1}{24}\omega_{+kl}J^{7kl}J^7_{ij} + \omega^{\mathbf{15}}_{ij+}, \end{aligned} \tag{254}$$

impose that

$$d\nu = -\frac{1}{16g^2}M_{ij}J^{7ij}J^7 + d\nu^{\mathbf{15}}, \tag{255}$$

together with

$$M = \frac{1}{8}M_{ij}J^{7ij}J^7. \tag{256}$$

Finally, $g\omega_{99+} = \frac{1}{3}g^7\omega_{+ij}J^{7ij}$ becomes

$$M_{ij} = \frac{1}{2(1+u^2)}J^7_{ij}. \tag{257}$$

Defining a new coordinate by

$$u = \tan\rho, \tag{258}$$

(257) becomes

$$\begin{aligned} \partial_\rho \hat{e}^1 &= \frac{1}{2}\hat{e}^2, \\ \partial_\rho \hat{e}^2 &= -\frac{1}{2}\hat{e}^1, \end{aligned} \tag{259}$$

and similarly for (e^3, e^4), (e^5, e^6), and (e^7, e^8). Hence

$$\begin{aligned} \hat{e}^1(\rho, x) &= \cos(\rho/2)\tilde{\hat{e}}^1(x) + \sin(\rho/2)\tilde{\hat{e}}^2(x), \\ \hat{e}^2(\rho, x) &= -\sin(\rho/2)\tilde{\hat{e}}^1(x) + \cos(\rho/2)\tilde{\hat{e}}^2(x), \end{aligned} \tag{260}$$

and similarly for the remaining pairs of achtbeins. As in case (i), the Killing spinors define a complex structure and a holomorphic four-form on the (conformally rescaled) base. The complex structure on the base is given by $J = \tilde{g}(x)J^7$. Both the metric and the complex structure are independent of ρ, since

$$\begin{aligned} \delta_{ij}\hat{e}^i\hat{e}^j &= \delta_{ij}\tilde{\hat{e}}^i\tilde{\hat{e}}^j, \\ \hat{e}^{12} + \hat{e}^{34} + \hat{e}^{56} + \hat{e}^{78} &= \tilde{\hat{e}}^{12} + \tilde{\hat{e}}^{34} + \tilde{\hat{e}}^{56} + \tilde{\hat{e}}^{78}. \end{aligned} \tag{261}$$

However, the holomorphic four-form, and hence the $SU(4)$ structure does depend on ρ; in terms of the $\tilde{\hat{e}}^i$, we have

$$\Omega = e^{-2i\rho}(\tilde{\hat{e}}^1 + i\tilde{\hat{e}}^2)(\tilde{\hat{e}}^3 + i\tilde{\hat{e}}^4)(\tilde{\hat{e}}^5 + i\tilde{\hat{e}}^6)(\tilde{\hat{e}}^7 + i\tilde{\hat{e}}^8). \tag{262}$$

Denoting the exterior derivative restricted to the base by $\tilde{d}$, if $\tilde{d}\Omega = 0$ at $\rho = 0$ then $\tilde{d}\Omega = 0$ for all ρ, as required. Finally, note that (255) implies that ν must satisfy

$$d\nu = -\frac{1}{4\tilde{g}^3}J + d\nu^{\mathbf{15}}. \tag{263}$$

We have now completely solved for the metric, Killing spinors and four-form in this case, so we will summarise the result. Defining $H^{-1/3} = \tilde{g}$, the Killing spinors are given by

$$\begin{aligned} f^7 &= z, \\ g &= \cos\rho H^{-1/3}(x), \\ g^7 &= \sin\rho H^{-1/3}(x). \end{aligned} \tag{264}$$

The metric is given by

$$ds^2 = H^{-2/3}(x)\Big[2d\rho(dv + \nu_M(x)dx^M) + \cos^2\rho dz^2\Big] + H^{1/3}(x)h_{MN}(x)dx^M dx^N, \tag{265}$$

where h_{MN} is a metric of $SU(4)$ holonomy and the complex structure and holomorphic four-form defined by the Killing spinors are

$$\begin{aligned} J &= \tilde{\hat{e}}^{12} + \tilde{\hat{e}}^{34} + \tilde{\hat{e}}^{56} + \tilde{\hat{e}}^{78}, \\ \Omega &= e^{-2i\rho}(\tilde{\hat{e}}^1 + i\tilde{\hat{e}}^2)(\tilde{\hat{e}}^3 + i\tilde{\hat{e}}^4)(\tilde{\hat{e}}^5 + i\tilde{\hat{e}}^6)(\tilde{\hat{e}}^7 + i\tilde{\hat{e}}^8), \end{aligned} \tag{266}$$

where $\tilde{\hat{e}}^i(x)$ are achtbeins for h. The one-form ν is required to satisfy

$$d\nu = -\frac{1}{4}HJ + d\nu^{\mathbf{15}}. \tag{267}$$

The solution evolves from a naked null singularity at $\rho = -\frac{\pi}{2}$ to a second naked null singularity at $\rho = \frac{\pi}{2}$. The flux is given by

$$\begin{aligned} F &= \cos\rho(dv + \nu) \wedge d\rho \wedge dz \wedge d(H^{-1}) + \cos\rho d\rho \wedge dz \wedge (-H^{-1}d\nu - J) \\ &+ \frac{1}{4!}F^{\mathbf{20}}_{ijkl}\tilde{e}^i \wedge \tilde{e}^j \wedge \tilde{e}^k \wedge \tilde{e}^l. \end{aligned} \tag{268}$$

The Bianchi identity imposes $F^{\mathbf{20}} = F^{\mathbf{20}}(x)$, $\tilde{d}F^{\mathbf{20}} = 0$. The $+-9$ component of the classical field equation is

$$\tilde{\nabla}^2 H = -\frac{1}{2 \times 4!}F^{\mathbf{20}}_{ijkl}F^{\mathbf{20}ijkl}, \tag{269}$$

where $\tilde{\nabla}^2$ denotes the Laplacian on the eight-manifold with metric h_{MN}, and here indices are raised with h^{MN}. All other field equations are identically satisfied. This class of solutions, together with those given above in case (i), exhausts all possibilities for $N = 4$ $SU(4)$ structures admitting a null Killing spinor.

References

[1] J. P. Gauntlett, D. Martelli, S. Pakis, and D. Waldram, *G-Structures and Wrapped NS5-Branes*, Commun.Math.Phys. 247 (2004) 421-445, hep-th/0205050.

[2] J. Gauntlett, J. Gutowski, C. Hull, S. Pakis and H. Reall, *All supersymmetric solutions of minimal supergravity in five dimensions*, Class. Quant. Grav. 20 (2003) 4587, hep-th/0209114.

[3] J. Gauntlett and J. Gutowski, *All supersymmetric solutions of minimal gauged supergravity in five dimensions*, Phys. Rev. D68, 105009 (2003), hep-th/0304064.

[4] M. Caldarelli and D. Klemm, *All supersymmetric solutions of N = 2, D = 4 gauged supergravity*, JHEP 0309 019 (2003, hep-th/0307022.

[5] S. L. Cacciatori, M. M. Caldarelli, D. Klemm and D. S. Mansi, *More on BPS solutions of N=2, D=4 gauged supergravity*, JHEP 0407 (2004) 061, hep-th/0406238.

[6] J. Gutowski, D. Martelli and H. Reall, *All supersymmetric solutions of minimal supergravity in six dimensions*, Class.Quant.Grav. 20 (2003) 5049-5078, hep-th/0306235.

[7] M. Cariglia and O. A. P. Mac Conamhna, *The general form of supersymmetric solutions of N=(1,0) U(1) and SU(2) gauged supergravities in six dimensions*, Class.Quant.Grav 21 (2004) 3171, hep-th/0402055.

[8] M. Cariglia and O. Mac Conamhna, *Timelike Killing spinors in seven dimensions*, Phys.Rev. D70 (2004) 125009, hep-th/0407127.

[9] J. Gauntlett and S. Pakis, *The geometry of D = 11 Killing spinors*, JHEP 0304 039 (2003), hep-th/0212008.

[10] J. P. Gauntlett, J. B. Gutowski and S. Pakis, *The Geometry of D=11 Null Killing Spinors*, JHEP 0312 (2003) 049, hep-th/0311112.

[11] D. Martelli and J. Sparks, *G-Structures, fluxes and calibrations in M-theory*, Phys. Rev. D68 085014 (2003), hep-th/0306225.

[12] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, *Supersymmetric AdS_5 solutions of M-theory*, Class.Quant.Grav. 21 (2004) 4335-4366, hep-th/0402153.

[13] A. Lukas and P.M. Saffin, *M-theory compactification, fluxes and AdS_4*, hep-th/0403235.

[14] G. Dall'Agata, *On supersymmetric solutions of type IIB supergravity with general fluxes*, Nucl.Phys. B695 (2004) 243-266, hep-th/0403220.

[15] A. R. Frey, *Notes on SU(3) Structures in Type IIB Supergravity*, JHEP 0406 (2004) 027, hep-th/0404107.

[16] K. Behrndt and C. Jeshek, *Fluxes in M-theory on 7-manifolds and G structures*, JHEP 0304 (2003) 002, hep-th/0302047.

[17] K. Behrndt and C. Jeschek, *Fluxes in M-theory on 7-manifolds: G-structures and Superpotential*, Nucl.Phys. B694 (2004) 99-114, hep-th/0311119.

[18] K. Behrndt, M. Cvetic and P. Gao, *General Type IIB Fluxes with SU(3) Structures*, hep-th/0502154.

[19] H. Lin, O. Lunin and J. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*,JHEP 0410 (2004) 025, hep-th/0409174.

[20] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, *A New Infinite Class of Sasaki-Einstein Manifolds*, hep-th/0403038.

[21] D. Martelli and J. Sparks, *Toric Geometry, Sasaki-Einstein Manifolds and a New Infinite Class of AdS/CFT Duals*, hep-th/0411238.

[22] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, *An Infinite Family of Superconformal Quiver Gauge Theories with Sasaki-Einstein Duals*, hep-th/0411264.

[23] J. B. Gutowski and H. S. Reall, *Supersymmetric AdS5 black holes*, JHEP 0402 (2004) 006, hep-th/0401042.

[24] J. B. Gutowski and H. S. Reall, *General supersymmetric AdS5 black holes*, JHEP 0404 (2004) 048, hep-th/0401129.

[25] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, *A supersymmetric black ring*, Phys.Rev.Lett. 93 (2004) 211302, hep-th/0407065.

[26] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, *Supersymmetric black rings and three-charge supertubes*, Phys.Rev. D71 (2005) 024033, hep-th/0408120.

[27] I. Bena and N. P. Warner, *One Ring to Rule Them All ... and in the Darkness Bind Them?*, hep-th/0408106.

[28] J. P. Gauntlett and J. B. Gutowski, *General Concentric Black Rings*, Phys.Rev. D71 (2005) 045002, hep-th/0408122.

[29] J. P. Gauntlett, *Classifying Supergravity Solutions*, hep-th/0501229.

[30] J. Figueroa-O'Farrill and G. Papadopoulos, *Maximally supersymmetric solutions of ten- and eleven-dimensional supergravities*, JHEP 0303 (2003) 048, hep-th/0211089.

[31] O. A. P. Mac Conamhna, *Refining G-structure classifications*, Phys.Rev. D70 (2004) 105024, hep-th/0408203.

[32] M. Cariglia and O. A. P. Mac Conamhna, *Null structure groups in eleven dimensions*, hep-th/0411079.

[33] M. Cariglia and O. A. P. Mac Conamhna, *Classification of supersymmetric spacetimes in eleven dimensions*, Phys. Rev. Lett. 94, 161601 (2005), hep-th/0412116.

[34] O. A. P. Mac Conamhna, in preparation.

[35] J. Gillard, U. Gran and G. Papadopoulos, *The spinorial geometry of supersymmetric backgrounds*, hep-th/0410155.

[36] U. Gran, J. Gutowski and G. Papadopoulos, *The spinorial geometry of supersymmetric IIB backgrounds*, hep-th/0501177.

[37] U. Gran, G. Papadopoulos and D. Roest, *Systematics of M-theory spinorial geometry*, hep-th/0503046.

[38] M. Cvetic, H. Lu and C. N. Pope, *Brane resolution through transgression*, Nucl.Phys. B600 (2001) 103-132, hep-th/0011023.

[39] C-M. Chen and J. F. Vazquez-Poritz, *Resolving the M2-brane*, hep-th/0403109.

[40] R. Clarkson, A.M. Ghezelbash and R.B. Mann, *New Reducible Membrane Solutions in D=11 Supergravity*, JHEP 0404 (2004) 063, hep-th/0404071.

[41] C. N. Gowdigere, D. Nemeschansky and N. P. Warner, *Supersymmetric Solutions with Fluxes from Algebraic Killing Spinors*, Adv.Theor.Math.Phys. 7 (2004) 787-806, hep-th/0306097.